On Positroids Induced by Unit Interval Orders

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Introduction

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Main Results:

1. We describe how every unit interval order induces a positroid (a special matroid coming from the totally nonnegative Grassmannian).
2. Then, we characterize the positroids arising in this way.
3. Specifically, there are Catalan-many unit interval orders, and we give a simple characterization of the decorated permutations of their associated positroids.
4. We describe bijections from the set of unit interval positroids (and unit interval orders) to the set of $2n$ length Dyck paths.

Outline

1. **Unit Interval Orders and Dyck Matrices**
2. **Unit Interval Positroids**
3. **Characterization of the Decorated Permutation**
4. **Interpretation of the f-vector of a Naturally Labeled Poset**

Definition of Unit Interval Orders

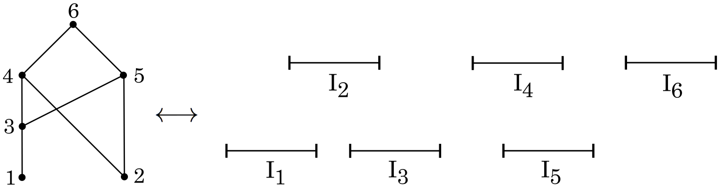
Definition

A poset P is a *unit interval order* if there exists a bijective map $i \mapsto [q_i, q_i + 1]$ from P to $S = \{[q_i, q_i + 1] \mid 1 \leq i \leq n,\ q_i \in \mathbb{R}\}$ such that for distinct $i, j \in P$,

$$i <_P j \quad \text{if and only if} \quad q_i + 1 < q_j. \tag{1}$$

We then say that S is an *interval representation* of P.

Example: A unit interval order and its interval representation:



Some Words on Unit Interval Orders

- Unit interval orders were introduced by Robert D. Luce in the context of economic sciences.
- They were used to axiomatize a class of utilities in the theory of preferences.
- Unit interval orders provide a mathematical framework for the theory of decision patterns.
- There are $\frac{1}{n+1}\binom{2n}{n}$ non-isomorphic unit interval orders on $[n]$.

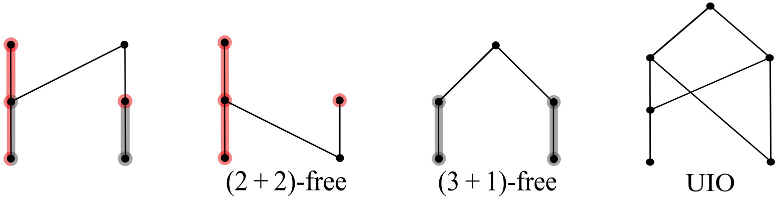
A Characterization of Unit Interval Orders

- A poset Q is an *induced* subposet of a poset P if there is an injective map $f : Q \to P$ such that $a <_Q b$ iff $f(a) <_P f(b)$.
- P is a *Q-free* poset if P does not contain any induced subposet isomorphic to Q.

Theorem (Scott-Suppes)

A poset is a unit interval order if and only if it is simultaneously $(\mathbf{3}+\mathbf{1})$-free and $(\mathbf{2}+\mathbf{2})$-free.

Examples:

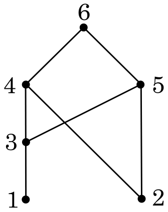


Natural and Canonical Labelings

Let P be a poset on $[n]$.

- P is *naturally labeled* if $i <_P j$ implies that $i \le j$ as integers.
- The *altitude* of P is the map $\alpha \colon P \to \mathbb{Z}$ defined by $i \mapsto |\Lambda_i| - |V_i|$.
- P is *canonically labeled* if $\alpha(i) < \alpha(j)$ implies $i < j$ (as integers).

Example: A canonically labeled poset:



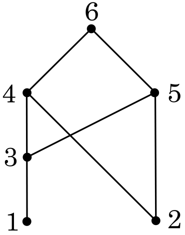
$\alpha(1) = -4$, $\alpha(2) = -3$, $\alpha(3) = -2, \alpha(4) = \alpha(5) = 2$, $\alpha(6) = 5$.

Antiadjacency Matrices of Labeled Posets

Definition (Antiadjacency Matrix)

If P is a poset on $[n]$, then the *antiadjacency matrix* of P is the $n \times n$ binary matrix $A = (a_{i,j})$ with $a_{i,j} = 0$ iff $i \neq j$ and $i <_P j$.

Example: A labeled poset and its antiadjacency matrix:



$$\longmapsto \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Dyck Matrices

A real square matrix is *totally nonnegative* if all its minors are ≥ 0.

Definition (Dyck Matrix)

A binary square matrix is said to be a *Dyck matrix* if its zero entries are above the main diagonal and its one entries are separated from its zero entries by a $2n$ length Dyck path supported on the main diagonal. We let $\mathcal{D}_n$ denote the set of Dyck matrices of size n.

Example: A 6×6 Dyck matrix and its Dyck path:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Observations:

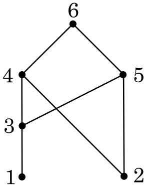
- Every Dyck matrix is totally nonnegative.
- $|\mathcal{D}_n| = \frac{1}{n+1}\binom{2n}{n}$, the n-th Catalan number.

Matrix Representation of Canonically Labeled Unit Interval Orders

Proposition (Reed-Skandera)

An n-labeled unit interval order is canonically labeled if and only if its antiadjacency matrix is a Dyck matrix.

Example:



$$\longmapsto \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Proposition (Chavez-G)

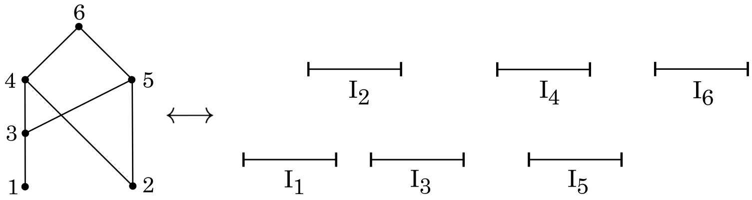
Let $\mathcal{U}_n$ be the set of non-isomorphic unit interval orders of size n. The map $\psi\colon \mathcal{U}_n \to \mathcal{D}_n$ assigning to each canonically labeled unit interval order its antiadjacency matrix is a bijection.

Interval Representations of Canonically Labeled Unit Interval Orders

Proposition (Chavez-G)

Let P be an n-labeled unit interval order. Then the labeling of P is canonical if and only if there exists an interval representation $\{[q_i, q_i + 1] \mid 1 \le i \le n\}$ of P such that $q_1 < \cdots < q_n$.

Example:



Unit Interval Positroids

Matroids

Definition (Matroid)

Let S be a finite set, and let β be a nonempty collection of subsets of S. The pair $M = (S, \beta)$ is a *matroid* if for all $A, B \in \beta$ and $a \in A \setminus B$, there exists $b \in B \setminus A$ such that $(A \setminus \{a\}) \cup \{b\} \in \beta$.

If $M = (S, \beta)$ is a matroid, then:

- the elements of β are said to be *bases* of M;
- the *rank* of M is the size of any basis (any two bases of M have the same cardinality).

Positroids

Definition: A matroid $([n], \beta)$ of rank d is *representable* if there is $X \in \mathrm{Mat}_{d,n}(\mathbb{R})$ with columns $X_1, \ldots, X_n$ such that $B \subseteq [n]$ belongs to β iff $\{X_i \mid i \in B\}$ is a basis for $\mathbb{R}^d$.

Notation: Let $\mathrm{Mat}^+_{d,n}(\mathbb{R})$ denote the set of all full rank $d \times n$ real matrices with nonnegative maximal minors.

Definition (Positroid)

A *positroid* on $[n]$ of rank d is a matroid that can be represented by a matrix in $\mathrm{Mat}^+_{d,n}(\mathbb{R})$.

Positroids (continuation)

Example: Consider the 3×6 real matrix

$$X = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 2 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

As X has rank 3 and all its maximal minors are nonnegative, it follows that $X \in \mathrm{Mat}^+_{3,6}(\mathbb{R})$. Therefore the matroid represented by X is a positroid $([6], \beta)$ whose collection of bases is

$$\beta = \big\{\{2, 4, 6\}, \{2, 5, 6\}\big\}.$$

Postnikov's Map

For $A = (a_{i,j}) \in \mathrm{Mat}_n(\mathbb{R})$, let $B = \phi(A) \in \mathrm{Mat}_{n,2n}(\mathbb{R})$, where

$$\begin{pmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n-1,1} & \dots & a_{n-1,n} \\ a_{n,1} & \dots & a_{n,n} \end{pmatrix} \overset{\phi}{\mapsto} \begin{pmatrix} 1 & \dots & 0 & 0 & \pm a_{n,1} & \dots & \pm a_{n,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & 0 & -a_{2,1} & \dots & -a_{2,n} \\ 0 & \dots & 0 & 1 & a_{1,1} & \dots & a_{1,n} \end{pmatrix}.$$

Notation: If $K \subseteq [2n]$ and $I, J \subseteq [n]$ so that $|K| = n$ and $|I| = |J|$,

- $\Delta_{I,J}(A)$ denotes the minor of A determined by the rows I and columns J;
- $\Delta_K(B)$ is the maximal minor of B determined by columns K.

Lemma (Postnikov)

Under this correspondence ϕ, $\Delta_{I,J}(A) = \Delta_{(n+1-[n]\setminus I)\cup(n+J)}(B)$ for all $I, J \subseteq [n]$ satisfying $|I| = |J|$.

Observation: $\phi(\mathcal{D}_n) \subset \mathrm{Mat}^+_{n,2n}(\mathbb{R})$.

Postnikov's Map (continuation)

Example: Consider the minor $\Delta_{I,J}(A)$ of the 3×3 real matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

where $I = J = \{2, 3\}$. Then

$$\phi(A) = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

and $(4 - [3] \setminus I) \cup (3 + J) = \{3, 5, 6\}$. Notice that

$$\Delta_{\{2,3\},\{2,3\}}(A) = 1 = \Delta_{\{3,5,6\}}(\phi(A)).$$

Unit Interval Positroids

Definition (Unit Interval Positroid)

For $D \in \mathcal{D}_n$, the positroid on $[2n]$ represented by $\phi(D)$ is called *unit interval positroid*. Let $\mathcal{P}_n$ denote the set of all unit interval positroids on $[2n]$.

Notation: Let $\rho\colon \mathrm{Mat}^{+}_{d,n}(\mathbb{R}) \to \mathcal{P}_n$ be the map sending each matrix in $\mathrm{Mat}^{+}_{d,n}(\mathbb{R})$ to the unit interval positroid it represents.

Theorem (Chavez-G)

The map $\rho \circ \phi\colon \mathcal{D}_n \to \mathcal{P}_n$ assigning to each Dyck matrix its corresponding unit interval positroid is a bijection.

Corollary

The map $\rho \circ \phi \circ \psi\colon \mathcal{U}_n \to \mathcal{P}_n$ is a bijection.

Decorated Permutation

Decorated Permutations

Definition (Decorated Permutation)

- A *decorated permutation* of $[n]$ is an element $\pi \in S_n$ whose fixed points j are marked either "clockwise"(denoted by $\pi(j) = \underline{j}$) or "counterclockwise" (denoted by $\pi(j) = \bar{j}$).
- A *weak excedance* (or *excedance*) of a decorated permutation $\pi \in S_n$ is an index $j \in [n]$ satisfying $j < \pi(j)$ or $\pi(j) = \bar{j}$.

Theorem (Postnikov)

There is natural bijection between rank d positroids on $[n]$ and decorated permutations of $[n]$ having exactly d excedances.

Decorated Permutations of Unit Interval Positroids

Decorated permutations of positroids in $\mathcal{P}_n$ are $2n$-cycles satisfying certain special properties.

Theorem (Chavez-G)

Decorated permutations associated to unit interval positroids on $[2n]$ are $2n$-cycles $(1\ j_1\ \dots\ j_{2n-1})$ satisfying the following two conditions:

1. *in the sequence $(1, j_1, \dots, j_{2n-1})$ the elements $1, \dots, n$ appear in increasing order while the elements $n+1, \dots, 2n$ appear in decreasing order;*
2. *for every $1 \leq k \leq 2n-1$, the set $\{1, j_1, \dots, j_k\}$ contains at least as many elements of the set $\{1, \dots, n\}$ as elements of the set $\{n+1, \dots, 2n\}$.*

Dyck Path Encoded in the Decorated Permutation of a Unit Interval Positroid

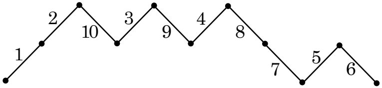
Example: The positroid represented by the 5×5 Dyck matrix

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

has decorated permutation

$$\pi = (1\ 2\ 10\ 3\ 9\ 4\ 8\ 7\ 5\ 6),$$

which has the following Dyck path encoded:

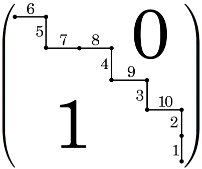


Decorated Permutation from Dyck Matrix

Theorem (Chavez-G)

If we number the n vertical steps of the Dyck path of $D \in \mathcal{D}_n$ from bottom to top with $1, \dots, n$ and the n horizontal steps from left to right with $n+1, \dots, 2n$, then we get the decorated permutation of the unit interval positroid induced by D by reading Dyck path of D in northwest direction.

Example: The decorated permutation π associated to the positroid represented by the 5×5 Dyck matrix D



can be read from the Dyck path of D, obtaining

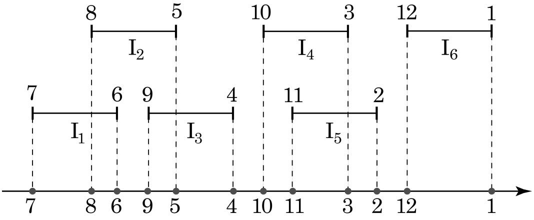
$$\pi = (1\ 2\ 10\ 3\ 9\ 4\ 8\ 7\ 5\ 6).$$

Decorated Permutation Read from Canonical Interval Representation

Theorem (Chavez-G)

Labeling the left and right endpoints of the intervals $[q_i, q_i + 1]$ by $n + i$ and $n + 1 - i$, respectively, we obtain the decorated permutation of the positroid induced by P by reading the label set $\{1, \dots, 2n\}$ from the real line from right to left.

Example: The decorated permutation (1 12 2 3 11 10 4 5 9 6 8 7) is obtained by reading the labels from right to left.



f-vector Interpretation

On f-vectors of Arbitrary (finite) Posets

Let P be a naturally labeled poset with antiadjacency matrix A_P.

Definition

If $|P| = n$, the *f-vector* of P is the sequence $f = (f_0, f_1, \ldots, f_{n-1})$, where f_k is the number of k-element chains of P.

Definition

A *valley Dyck path* of A_P is a Dyck path drawn inside A_P that has its endpoints and all its valleys on the main diagonal and all its peaks in positions (i, j) such that $a_{i,j} = 0$.

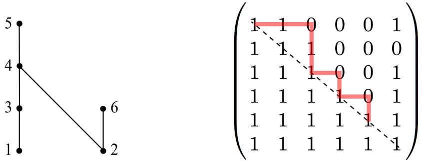


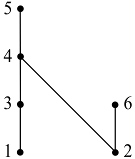
Figure: A poset and its antiadjacency matrix showing a 3-peak valley Dyck path.

Characterization of f-vectors

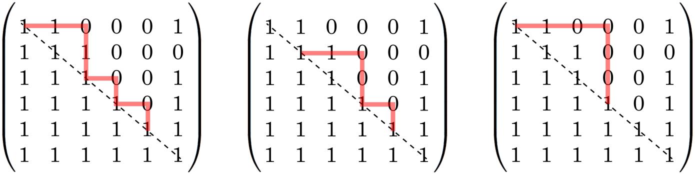
Proposition (Chavez-G)

The entries of the f-vector of P are $f_0 = n$ and f_k equals the number of valley Dyck paths of A_P having exactly k peaks.

Example:



$f = (6, 9, 4, 1, 0, 0)$



$1 < 3 < 4 < 5$ $2 < 4 < 5$ $1 < 4$

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