Dyck Paths and Positroids from Unit Interval Orders

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Introduction

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Motivation: Most of the initial questions motivating this project were provided by Alejandro Morales.

Main Results:

1. We propose a natural bijection from the set of unit interval orders on $[n]$ to a special subset $\mathcal{P}_n$ of positroids on $[2n]$ of rank $[n]$.
2. We also characterize the decorated permutations associated to the positroids in $\mathcal{P}_n$.
3. We interpret the f-vector of a naturally labeled poset in terms of its antiadjacency matrix.

Outline

Matroids

Definition (Matroid)

Let S be a finite set, and let β be a nonempty collection of subsets of S. The pair $M = (S, \beta)$ is a *matroid* if for all $A, B \in \beta$ and $a \in A \setminus B$, there exists $b \in B \setminus A$ such that $(A \setminus \{a\}) \cup \{b\} \in \beta$.

If $M = (S, \beta)$ is a matroid, then:

- the elements of β are said to be *bases* of M;
- the *rank* of M, denoted by $r(M)$, is the size of any basis (any two bases of M have the same cardinality).

Positroids

Definition: A matroid $([n], \beta)$ of rank d is *representable* if there is $X \in M_{d\times n}(\mathbb{R})$ with columns $X_1, \dots, X_n$ such that $B \subseteq [n]$ belongs to β iff $\{X_i \mid i \in B\}$ is a basis for $\mathbb{R}^d$.

Notation: Let $\mathrm{Mat}^+_{d,n}$ denote the set of all full rank $d \times n$ real matrices with nonnegative maximal minors.

Definition (Positroid)

A *positroid* on $[n]$ of rank d is a matroid that can be represented by a matrix in $\mathrm{Mat}^+_{d,n}$.

Positroids (continuation)

Example: Consider the 3×6 real matrix

$$X = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 2 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

As X has rank 3 and all its minors are nonnegative, it follows that $X \in \mathrm{Mat}^+_{3,6}$. Therefore the matroid represented by X is a positroid $([6], \beta)$ whose collection of bases is

$$\beta = \big\{\{2,4,6\},\{2,5,6\}\big\}.$$

Decorated Permutations

Definition: The *i-order* on $[n]$ is given by

$$i <_i \cdots <_i n <_i 1 <_i \cdots <_i i-1.$$

Definition (Decorated Permutation)

- A *decorated permutation* of $[n]$ is an element $\pi \in S_n$ whose fixed points j are marked either "clockwise"(denoted by $\pi(j) = \underline{j}$) or "counterclockwise" (denoted by $\pi(j) = \bar{j}$).
- A *weak i-excedance* (or *excedance*) of a decorated permutation $\pi \in S_n$ is an index $j \in [n]$ satisfying $j <_i \pi(j)$ or $\pi(j) = \bar{j}$.

Theorem (Postnikov)

There is natural bijection between rank d positroids on $[2n]$ and decorated permutations of $[n]$ having exactly d excedances.

Decorated Permutations (continuation)

Lemma

Let $X \in Mat^+_{d,n}$, and let π be the decorated permutation associated to the positroid represented by X.

- *For $i \in \{1, \dots, n\}$, $\pi(i)$ equals the minimum $j \in [n]$ with respect to the i-order such that $B_i \in span(B_{i+1}, \dots, B_j)$.*
- *A fixed point j of π is marked clockwise iff B_j is zero.*

Example: We have already seen that the 3×6 real matrix

$$X = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 2 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \in \mathrm{Mat}^+_{3,6}$$

represents the positroid $P = \big([6], \{\{2,4,6\}, \{2,5,6\}\}\big)$. The decorated permutation associated to P is

$$\pi = (\underline{1})\,(\overline{2})\,(\underline{3})\,(4\ 5)\,(\overline{6}).$$

Dyck Matrices

A square matrix is *totally nonnegative* if all its minors are ≥ 0.

Definition (Dyck Matrix)

A binary square matrix is said to be a *Dyck matrix* if its zero entries are above the main diagonal and its one entries are separated from its zero entries by a Dyck path supported on the main diagonal. We let $\mathcal{D}_n$ denote the set of Dyck matrices of size n.

Example: A 6×6 Dyck matrix and its Dyck path:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Observations:

- Every Dyck matrix is totally nonnegative.
- $|\mathcal{D}_n| = \frac{1}{n+1}\binom{2n}{n}$, the n-th Catalan number.

Postnikov's Map

In Postnikov's *Total positivity, Grassmannians, and networks* we find the next lemma:

Lemma

For an $n \times n$ real matrix $A = (a_{i,j})$, consider the $n \times 2n$ matrix $B = \phi(A)$, where

$$\begin{pmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n-1,1} & \dots & a_{n-1,n} \\ a_{n,1} & \dots & a_{n,n} \end{pmatrix} \overset{\phi}{\mapsto} \begin{pmatrix} 1 & \dots & 0 & 0 & \pm a_{n,1} & \dots & \pm a_{n,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & 0 & -a_{2,1} & \dots & -a_{2,n} \\ 0 & \dots & 0 & 1 & a_{1,1} & \dots & a_{1,n} \end{pmatrix}.$$

Under this correspondence, $\Delta_{I,J}(A) = \Delta_{(n+1-[n]\setminus I)\cup(n+J)}(B)$ for all $I, J \subseteq [n]$ satisfying $|I| = |J|$ (here $\Delta_{I,J}(A)$ is the minor of A determined by the rows I and columns J, and $\Delta_K(B)$ is the maximal minor of B determined by columns K).

Postnikov's Map (continuation)

Example: Consider the minor $\Delta_{I,J}(A)$ of the 3×3 real matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 3 \\ 3 & 0 & 5 \end{pmatrix}$$

determined by the set of index row $I = \{2,3\}$ and the set of index column $J = \{1,2\}$. As $(4 - [3] \setminus I) \cup (3 + J) = \{3,4,5\}$, it follows that $\Delta_{I,J}(A) = \Delta_{\{3,4,5\}}(\phi(A))$, where

$$\phi(A) = \begin{pmatrix} 1 & 0 & 0 & 3 & 0 & 5 \\ 0 & 1 & 0 & -1 & -2 & -3 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

Note that $\Delta_{I,J}(A) = \Delta_{\{3,4,5\}}(\phi(A)) = -6$.

Unit Interval Positroids

Remark: $\phi(\mathcal{D}_n) \subset \mathrm{Mat}^+_{n,2n}$.

Definition (Unit Interval Positroid)

For $D \in \mathcal{D}_n$, the positroid on $[2n]$ represented by $\phi(D)$ is called *unit interval positroid*. Let $\mathcal{P}_n$ denote the set of all unit interval positroids on $[2n]$.

Comment on the terminology: We shall see later that there is a natural bijection between unit interval orders and Dyck matrices, which justifies the terminology in the definition above.

Theorem (C-G)

The map $\varphi \colon \mathcal{D}_n \to \mathcal{P}_n$ assigning to each Dyck matrix its corresponding unit interval positroid is a bijection.

Corollary: There are $\frac{1}{n+1}\binom{2n}{n}$ unit interval positroids on $[2n]$.

Decorated Permutations of Unit Interval Positroids

Decorated permutations of positroids in $\mathcal{P}_n$ are $2n$-cycles satisfying certain special properties.

Theorem (C-G)

Decorated permutations associated to unit interval positroids on $[2n]$ are $2n$-cycles $(1\ j_1\ \ldots\ j_{2n-1})$ satisfying the following two conditions:

1. *in the sequence $(1, j_1, \ldots, j_{2n-1})$ the elements $1, \ldots, n$ appear in increasing order while the elements $n+1, \ldots, 2n$ appear in decreasing order;*
2. *for every $1 \leq k \leq 2n-1$, the set $\{1, j_1, \ldots, j_k\}$ contains at least as many elements of the set $\{1, \ldots, n\}$ as elements of the set $\{n+1, \ldots, 2n\}$.*

Dyck Path Encoded in the Decorated Permutation of a Unit Interval Positroid

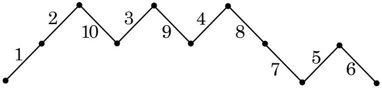
Example: The positroid represented by the 5×5 Dyck matrix

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

has decorated permutation

$$\pi = (1\ 2\ 10\ 3\ 9\ 4\ 8\ 7\ 5\ 6),$$

which has the following Dyck path encoded:

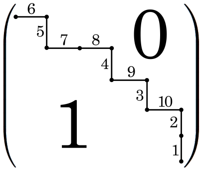


Decorated Permutation from Dyck Matrix

Theorem (C-G)

If we number the n vertical steps of the Dyck path of $D \in \mathcal{D}_n$ from bottom to top with $1, \ldots, n$ and the n horizontal steps from left to right with $n+1, \ldots, 2n$, then we get the decorated permutation of the unit interval positroid induced by D by reading Dyck path of D in northwest direction.

Example: The decorated permutation π associated to the positroid represented by the 5×5 Dyck matrix D



can be read from the Dyck path of D, obtaining

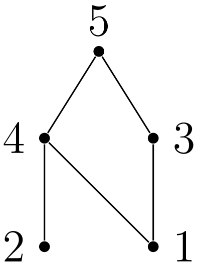
$$\pi = (1\ 2\ 10\ 3\ 9\ 4\ 8\ 7\ 5\ 6).$$

Unit Interval Orders

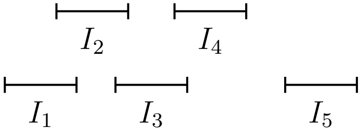
Definition

A poset P is a *unit interval order* if there exists a bijective map $i \mapsto [q_i, q_i + 1]$ from P to $S = \{[q_i, q_i + 1] \mid 1 \leq i \leq n,\, q_i \in \mathbb{R}\}$ such that for distinct $i, j \in P$, $i <_P j$ if and only if $q_i + 1 < q_j$. We then say that S is an *interval representation* of P.

Example:



$\longmapsto$



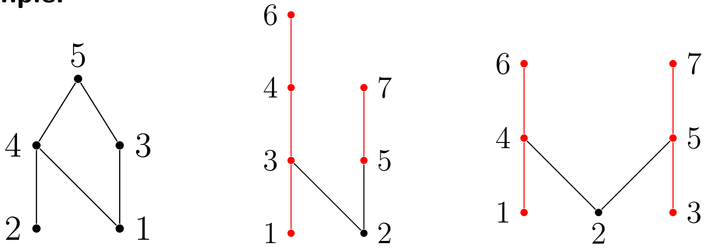
Unit Interval Orders

- A subset Q is an *induced* subposet of P if there is an injective map $f\colon Q \to P$ such that $r <_Q s$ if and only if $f(r) <_P f(s)$.
- P is a *Q-free* poset if P does not contain any induced subposet isomorphic to Q.

Theorem

A poset is a unit interval order if and only if it is simultaneously $(\mathbf{3}+\mathbf{1})$-free and $(\mathbf{2}+\mathbf{2})$-free.

Example:



Natural and Canonical Labelings

Let P be a poset on $[n]$.

- P is *naturally labeled* if $i <_P j$ implies that $i \leq j$ as integers.
- The *altitude* of P is the map $\alpha\colon P \to \mathbb{Z}$ defined by $i \mapsto |\Lambda_i| - |\mathsf{V}_i|$.
- P is *canonically labeled* if $\alpha(i) < \alpha(j)$ implies $i < j$ (as integers).
- An n-labeled poset if it respects altitude.

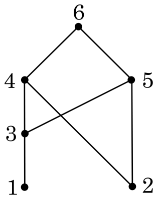


Figure: A canonically labeled poset on [6].

Antiadjacency Matrices of Labeled Posets

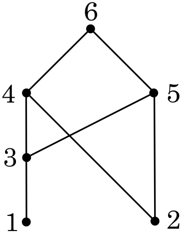
Definition (Antiadjacency Matrix)

If P is a poset $[n]$, then the *antiadjacency matrix* of P is the $n \times n$ binary matrix $A = (a_{i,j})$ with $a_{i,j} = 0$ iff $i \neq j$ and $i <_P j$.

Proposition

An n-labeled unit interval order is canonically labeled if and only if its antiadjacency matrix is a Dyck matrix.

Example:



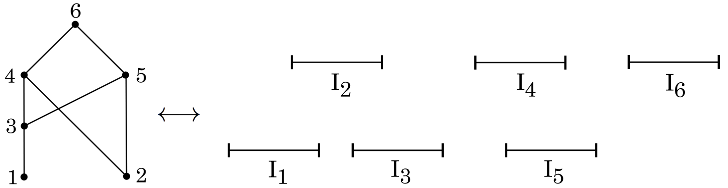
$$\longmapsto \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Canonical Interval Representation

Proposition (C-G)

Let P be a unit interval order on $[n]$. Then the labeling of P is canonical if and only if there exists an interval representation $\{[q_i, q_i+1] \mid 1 \leq i \leq n\}$ of P such that $q_1 < \cdots < q_n$.

Example:



Bijection between $\mathcal{U}_n$ and $\mathcal{P}_n$

Recall which are the maps ρ, ϕ...

Proposition (C-G)

For every n, the map $\varphi\colon \mathcal{U}_n \to \mathcal{D}_n$ assigning to each unit interval order its associated matrix is a bijection.

- Recall that $\mathrm{Mat}^+_{d,n}$ denotes the set of all full rank $d \times n$ real matrices with nonnegative maximal minors.
- Let ρ: $\mathrm{Mat}^+_{d,n} \to \mathcal{P}_n$ denote the map mapping each matrix in $\mathrm{Mat}^+_{d,n}$ the positroid it represents.

Theorem (C-G)

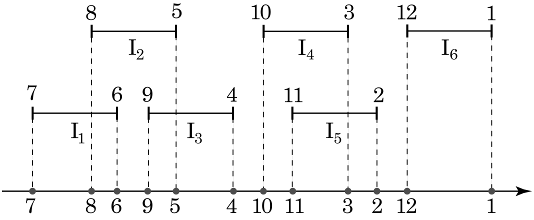
The map $\rho \circ \phi \circ \varphi\colon \mathcal{U}_n \to \mathcal{P}_n$ is a bijection, where ϕ is the map in the Postnikov's lemma.

Decorated Permutation Read from Canonical Interval Representation

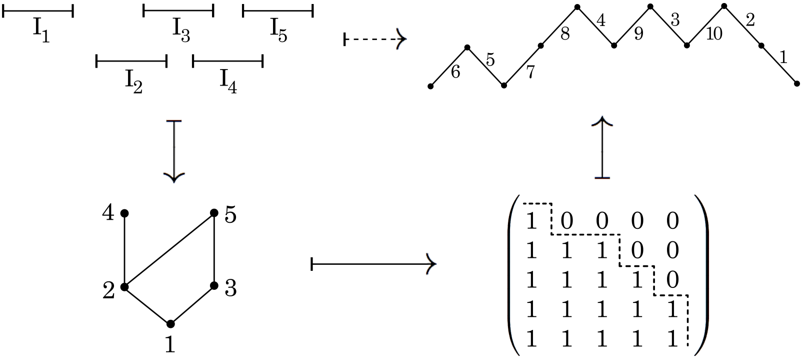
Theorem (C-G)

Labeling the left and right endpoints of the intervals $[q_i, q_i + 1]$ by $n + i$ and $n + 1 - i$, respectively, we obtain the decorated permutation of the positroid induced by P by reading the label set $\{1, \ldots, 2n\}$ from the real line from right to left.

Example: The decorated permutation $(1\ 12\ 2\ 3\ 11\ 10\ 4\ 5\ 9\ 6\ 8\ 7)$ is obtained by reading the labels from right to left.



Invervals to Positroids



On f-vectors of Unit Interval Orders

Skandera and Reed posed the problem:

Characterize the f-vectors of unit interval orders.

With this in mind, we provide a description of the entries of the f-vector of any naturally labeled poset as the number of special Dyck paths arising from its associated antiadjacency matrix.

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

On f-vectors of Naturally Labeled Posets

Definition

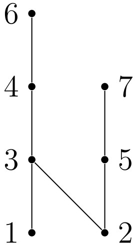
Let P be a naturally labeled poset on n elements. Define a *k-chain* in P to be the collection of $k+1$ elements of P such that $x_1 <_P x_2 <_P \cdots <_P x_{k+1}$.

Definition (f-vector)

The f-vector of a naturally labeled poset is a sequence of integers such that f_k is the number of k-chains in P.

On f-vectors of Naturally Labeled Posets

Example: The naturally ordered poset P



has f-vector $f = (7, 12, 8, 2, 0, 0, 0)$.

On f-vectors of Naturally Labeled Posets (cont.)

Here is a description of the f-vector.

Theorem (C-G)

Let P be a naturally labeled poset on $[n]$. Then $f_0 = n$ and f_k is the number of Dyck paths with k 0-peaks realized on the antiadjacency matrix of P for $0 < k < n$.

Note: this theorem holds for all naturally labeled posets, so in particular it holds for unit interval orders.

On f-vectors of Naturally Labeled Posets

Example: Dyck paths used to calculate $f = (7, 12, 8, 2, 0, 0, 0)$:

f_1

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

f_2

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

f_3

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

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