# Minimal presentations of shifted numerical monoids


Christopher O'Neill

Texas A&M University

*coneill@math.tamu.edu*

Joint with Rebecca Conaway*, Felix Gotti, Jesse Horton*,
Roberto Pelayo, Mesa Williams*, and Brian Wissman

* = undergraduate student


March 6, 2016

# Numerical monoids

**Definition**

A *numerical monoid* $S$ is an **additive** submonoid of $\mathbb{N}$ with $|\mathbb{N} \setminus S| < \infty$.

# Numerical monoids

## Definition

A *numerical monoid* $S$ is an **additive** submonoid of $\mathbb{N}$ with $|\mathbb{N} \setminus S| < \infty$.

## Example

$McN = \langle 6, 9, 20 \rangle = \{0, 6, 9, 12, 15, 18, 20, 21, \ldots\}$.

# Numerical monoids

**Definition**

A *numerical monoid* $S$ is an **additive** submonoid of $\mathbb{N}$ with $|\mathbb{N} \setminus S| < \infty$.

**Example**

$McN = \langle 6, 9, 20 \rangle = \{0, 6, 9, 12, 15, 18, 20, 21, \ldots\}$. “McNugget Monoid”

# Numerical monoids

**Definition**

A *numerical monoid* $S$ is an **additive** submonoid of $\mathbb{N}$ with $|\mathbb{N} \setminus S| < \infty$.

**Example**

$McN = \langle 6, 9, 20 \rangle = \{0, 6, 9, 12, 15, 18, 20, 21, \ldots\}$. "McNugget Monoid"
Factorizations:

$$60 \;=$$

# Numerical monoids

## Definition

A *numerical monoid* $S$ is an **additive** submonoid of $\mathbb{N}$ with $|\mathbb{N} \setminus S| < \infty$.

## Example

$McN = \langle 6, 9, 20 \rangle = \{0, 6, 9, 12, 15, 18, 20, 21, \ldots\}$. "McNugget Monoid"
Factorizations:

$$60 \quad = \quad 7(6) + 2(9)$$

# Numerical monoids

## Definition

A *numerical monoid* $S$ is an **additive** submonoid of $\mathbb{N}$ with $|\mathbb{N} \setminus S| < \infty$.

## Example

$McN = \langle 6, 9, 20 \rangle = \{0, 6, 9, 12, 15, 18, 20, 21, \ldots\}$. "McNugget Monoid"

Factorizations:

$$\begin{aligned} 60 &= 7(6) + 2(9) \\ &= \qquad\qquad 3(20) \end{aligned}$$

# Numerical monoids

## Definition

A *numerical monoid* $S$ is an **additive** submonoid of $\mathbb{N}$ with $|\mathbb{N} \setminus S| < \infty$.

## Example

$McN = \langle 6, 9, 20 \rangle = \{0, 6, 9, 12, 15, 18, 20, 21, \ldots\}$. "McNugget Monoid"

Factorizations:

$$\begin{aligned} 60 &= 7(6) + 2(9) && \rightsquigarrow && (7, 2, 0) \\ &= 3(20) && \rightsquigarrow && (0, 0, 3) \end{aligned}$$

# To shift a numerical monoid...

Fix $S = \langle r_1, \ldots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle.$$

# To shift a numerical monoid...

Fix $S = \langle r_1, \ldots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle.$$

Catenary degree $\mathsf{c}(M_n)$: measures spread of factorizations in $M_n$.

# To shift a numerical monoid…

Fix $S = \langle r_1, \ldots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle.$$

Catenary degree $\mathsf{c}(M_n)$: measures spread of factorizations in $M_n$.

$M_n = \langle n, n + 6, n + 9, n + 20 \rangle$:

# To shift a numerical monoid...

Fix $S = \langle r_1, \ldots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle.$$

Catenary degree $\mathsf{c}(M_n)$: measures spread of factorizations in $M_n$.

$M_n = \langle n, n + 6, n + 9, n + 20 \rangle$:

# To shift a numerical monoid...

Fix $S = \langle r_1, \ldots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle.$$

Catenary degree $\mathsf{c}(M_n)$: measures spread of factorizations in $M_n$.

$M_n = \langle n, n + 6, n + 9, n + 20 \rangle$:

$$\mathsf{c}(M_n) \text{ is periodic-linear (quasilinear) for } n \geq 126.$$

# To shift a numerical monoid...

Fix $S = \langle r_1, \ldots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle.$$

# To shift a numerical monoid...

Fix $S = \langle r_1, \ldots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle.$$

Delta set $\Delta(M_n)$: successive factorization length differences in $M_n$.

# To shift a numerical monoid...

Fix $S = \langle r_1, \ldots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle.$$

Delta set $\Delta(M_n)$: successive factorization length differences in $M_n$.

**Theorem (Chapman-Kaplan-Lemburg-Niles-Zlogar, 2014)**

*The delta set $\Delta(M_n)$ is singleton for $n \gg 0$.*

# To shift a numerical monoid...

Fix $S = \langle r_1, \ldots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle.$$

Delta set $\Delta(M_n)$: successive factorization length differences in $M_n$.

**Theorem (Chapman-Kaplan-Lemburg-Niles-Zlogar, 2014)**

*The delta set $\Delta(M_n)$ is singleton for $n \gg 0$.*

$M_n = \langle n, n + 6, n + 9, n + 20 \rangle$:

# To shift a numerical monoid...

Fix $S = \langle r_1, \dots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \dots, n + r_k \rangle.$$

Delta set $\Delta(M_n)$: successive factorization length differences in $M_n$.

**Theorem (Chapman-Kaplan-Lemburg-Niles-Zlogar, 2014)**

*The delta set $\Delta(M_n)$ is singleton for $n \gg 0$.*

$M_n = \langle n, n + 6, n + 9, n + 20 \rangle$:

$$\Delta(M_n) = \{1\} \text{ for all } n \geq 48$$

# To shift a numerical monoid…

Fix $S = \langle r_1, \ldots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle.$$

# To shift a numerical monoid…

Fix $S = \langle r_1, \ldots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle.$$

Betti numbers $\beta_i(M_n)$: Betti numbers of the defining toric ideal $I_{M_n}$.

# To shift a numerical monoid...

Fix $S = \langle r_1, \ldots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle.$$

Betti numbers $\beta_i(M_n)$: Betti numbers of the defining toric ideal $I_{M_n}$.

**Theorem (Vu, 2014)**

*The Betti numbers of $M_n$ are eventually $r_k$-periodic in $n$.*

# To shift a numerical monoid...

Fix $S = \langle r_1, \ldots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle.$$

Betti numbers $\beta_i(M_n)$: Betti numbers of the defining toric ideal $I_{M_n}$.

## Theorem (Vu, 2014)

*The Betti numbers of $M_n$ are eventually $r_k$-periodic in $n$.*

$M_n = \langle n, n + 6, n + 9, n + 20 \rangle$: Graded degrees for $\beta_0(M_n)$

# To shift a numerical monoid…

Fix $S = \langle r_1, \ldots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle.$$

# To shift a numerical monoid…

Fix $S = \langle r_1, \ldots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle.$$

Observations:

# To shift a numerical monoid...

Fix $S = \langle r_1, \ldots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle.$$

Observations:

- Known: the Betti numbers $n \mapsto \beta_i(M_n)$ are eventually $r_k$-periodic.

# To shift a numerical monoid...

Fix $S = \langle r_1, \ldots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle.$$

Observations:

- Known: the Betti numbers $n \mapsto \beta_i(M_n)$ are eventually $r_k$-periodic.
- Known: the function $n \mapsto \Delta(M_n)$ is eventually singleton.

# To shift a numerical monoid...

Fix $S = \langle r_1, \ldots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle.$$

Observations:

- Known: the Betti numbers $n \mapsto \beta_i(M_n)$ are eventually $r_k$-periodic.
- Known: the function $n \mapsto \Delta(M_n)$ is eventually singleton.
- Observed: the function $n \mapsto \mathsf{c}(M_n)$ is eventually $r_k$-quasilinear.

# To shift a numerical monoid...

Fix $S = \langle r_1, \ldots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle.$$

Observations:

- Known: the Betti numbers $n \mapsto \beta_i(M_n)$ are eventually $r_k$-periodic.
- Known: the function $n \mapsto \Delta(M_n)$ is eventually singleton.
- Observed: the function $n \mapsto \mathsf{c}(M_n)$ is eventually $r_k$-quasilinear.

Underlying cause:

# To shift a numerical monoid...

Fix $S = \langle r_1, \ldots, r_k \rangle \subset (\mathbb{N}, +)$, and let

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle.$$

Observations:

- Known: the Betti numbers $n \mapsto \beta_i(M_n)$ are eventually $r_k$-periodic.
- Known: the function $n \mapsto \Delta(M_n)$ is eventually singleton.
- Observed: the function $n \mapsto \mathsf{c}(M_n)$ is eventually $r_k$-quasilinear.

Underlying cause: minimal presentations!

# Kernel congruences and minimal presentations

Let $S = \langle r_1, \ldots, r_k \rangle$.

$$n = a_1 r_1 + \cdots + a_k r_k \qquad \leftrightsquigarrow \qquad \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{N}^k$$

# Kernel congruences and minimal presentations

Let $S = \langle r_1, \ldots, r_k \rangle$.

$$n = a_1 r_1 + \cdots + a_k r_k \qquad \leftrightsquigarrow \qquad \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{N}^k$$

Factorization homomorphism:

$$\begin{aligned} \pi : \mathbb{N}^k &\longrightarrow \langle r_1, \ldots, r_k \rangle \\ \mathbf{a} &\longmapsto a_1 r_1 + \cdots + a_k r_k \end{aligned}$$

# Kernel congruences and minimal presentations

Let $S = \langle r_1, \ldots, r_k \rangle$.

$$n = a_1 r_1 + \cdots + a_k r_k \qquad \leftrightsquigarrow \qquad \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{N}^k$$

Factorization homomorphism:

$$\begin{array}{rcl} \pi : \mathbb{N}^k & \longrightarrow & \langle r_1, \ldots, r_k \rangle \\ \mathbf{a} & \longmapsto & a_1 r_1 + \cdots + a_k r_k \end{array}$$

## Definition

The *kernel* $\ker \pi$ is the relation $\sim$ on $\mathbb{N}^k$ with $\mathbf{a} \sim \mathbf{b}$ whenever

$$\pi(\mathbf{a}) = \pi(\mathbf{b})$$

# Kernel congruences and minimal presentations

Let $S = \langle r_1, \ldots, r_k \rangle$.

$$n = a_1 r_1 + \cdots + a_k r_k \qquad \leftrightsquigarrow \qquad \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{N}^k$$

Factorization homomorphism:

$$\begin{aligned} \pi : \mathbb{N}^k &\longrightarrow \langle r_1, \ldots, r_k \rangle \\ \mathbf{a} &\longmapsto a_1 r_1 + \cdots + a_k r_k \end{aligned}$$

## Definition

The *kernel* $\ker \pi$ is the relation $\sim$ on $\mathbb{N}^k$ with $\mathbf{a} \sim \mathbf{b}$ whenever

$$\pi(\mathbf{a}) = \pi(\mathbf{b})$$

$\ker \pi$ is a *congruence*: an equivalence relation

$$\mathbf{a} \sim \mathbf{a}$$
$$\mathbf{a} \sim \mathbf{b} \Rightarrow \mathbf{b} \sim \mathbf{a}$$
$$\mathbf{a} \sim \mathbf{b} \text{ and } \mathbf{b} \sim \mathbf{c} \Rightarrow \mathbf{a} \sim \mathbf{c}$$

# Kernel congruences and minimal presentations

Let $S = \langle r_1, \ldots, r_k \rangle$.

$$n = a_1 r_1 + \cdots + a_k r_k \qquad \leftrightsquigarrow \qquad \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{N}^k$$

Factorization homomorphism:

$$\begin{array}{rcl} \pi : \mathbb{N}^k & \longrightarrow & \langle r_1, \ldots, r_k \rangle \\ \mathbf{a} & \longmapsto & a_1 r_1 + \cdots + a_k r_k \end{array}$$

## Definition

The *kernel* $\ker \pi$ is the relation $\sim$ on $\mathbb{N}^k$ with $\mathbf{a} \sim \mathbf{b}$ whenever

$$\pi(\mathbf{a}) = \pi(\mathbf{b})$$

$\ker \pi$ is a *congruence*: an equivalence relation

$$\mathbf{a} \sim \mathbf{a}$$
$$\mathbf{a} \sim \mathbf{b} \Rightarrow \mathbf{b} \sim \mathbf{a}$$
$$\mathbf{a} \sim \mathbf{b} \text{ and } \mathbf{b} \sim \mathbf{c} \Rightarrow \mathbf{a} \sim \mathbf{c}$$

that is closed under *translation*.

$$\mathbf{a} \sim \mathbf{b} \Rightarrow \mathbf{a} + \mathbf{c} \sim \mathbf{b} + \mathbf{c}$$

# Kernel congruences and minimal presentations

Let $S = \langle r_1, \ldots, r_k \rangle$.

$$n = a_1 r_1 + \cdots + a_k r_k \qquad \rightsquigarrow \qquad \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{N}^k$$

Factorization homomorphism:

$$\begin{aligned} \pi : \mathbb{N}^k &\longrightarrow \langle r_1, \ldots, r_k \rangle \\ \mathbf{a} &\longmapsto a_1 r_1 + \cdots + a_k r_k \end{aligned}$$

# Kernel congruences and minimal presentations

Let $S = \langle r_1, \ldots, r_k \rangle$.

$$n = a_1 r_1 + \cdots + a_k r_k \qquad \leftrightsquigarrow \qquad \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{N}^k$$

Factorization homomorphism:

$$\begin{array}{rcl} \pi : \mathbb{N}^k & \longrightarrow & \langle r_1, \ldots, r_k \rangle \\ \mathbf{a} & \longmapsto & a_1 r_1 + \cdots + a_k r_k \end{array}$$

## Definition

A *minimal presentation* $\rho$ of $S$ is a minimal subset $\rho \subset \ker \pi$ whose reflexive, symmetric, transitive, and translation closure equals $\ker \pi$.

# Kernel congruences and minimal presentations

Let $S = \langle r_1, \ldots, r_k \rangle$.

$$n = a_1 r_1 + \cdots + a_k r_k \qquad \leftrightsquigarrow \qquad \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{N}^k$$

Factorization homomorphism:

$$\begin{array}{rcl} \pi : \mathbb{N}^k & \longrightarrow & \langle r_1, \ldots, r_k \rangle \\ \mathbf{a} & \longmapsto & a_1 r_1 + \cdots + a_k r_k \end{array}$$

**Definition**

A *minimal presentation* $\rho$ of $S$ is a minimal subset $\rho \subset \ker \pi$ whose reflexive, symmetric, transitive, and translation closure equals $\ker \pi$.

$S = \langle 6, 9, 20 \rangle$:

# Kernel congruences and minimal presentations

Let $S = \langle r_1, \ldots, r_k \rangle$.

$$n = a_1 r_1 + \cdots + a_k r_k \qquad \leftrightsquigarrow \qquad \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{N}^k$$

Factorization homomorphism:

$$\begin{array}{rcl} \pi : \mathbb{N}^k & \longrightarrow & \langle r_1, \ldots, r_k \rangle \\ \mathbf{a} & \longmapsto & a_1 r_1 + \cdots + a_k r_k \end{array}$$

## Definition

A *minimal presentation* $\rho$ of $S$ is a minimal subset $\rho \subset \ker \pi$ whose reflexive, symmetric, transitive, and translation closure equals $\ker \pi$.

$S = \langle 6, 9, 20 \rangle$: $\rho = \{((3,0,0),(0,2,0)),((7,2,0),(0,0,3))\} \subset \ker \pi$

# Kernel congruences and minimal presentations

Let $S = \langle r_1, \ldots, r_k \rangle$.

$$n = a_1 r_1 + \cdots + a_k r_k \qquad \leftrightsquigarrow \qquad \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{N}^k$$

Factorization homomorphism:

$$\begin{array}{rcl} \pi : \mathbb{N}^k & \longrightarrow & \langle r_1, \ldots, r_k \rangle \\ \mathbf{a} & \longmapsto & a_1 r_1 + \cdots + a_k r_k \end{array}$$

## Definition

A *minimal presentation* $\rho$ of $S$ is a minimal subset $\rho \subset \ker \pi$ whose reflexive, symmetric, transitive, and translation closure equals $\ker \pi$.

$S = \langle 6, 9, 20 \rangle$: $\rho = \{((3,0,0),(0,2,0)), ((7,2,0),(0,0,3))\} \subset \ker \pi$

$\pi^{-1}(60)$:

bettiexample.pdf

# Kernel congruences and minimal presentations

Let $S = \langle r_1, \ldots, r_k \rangle$.

$$n = a_1 r_1 + \cdots + a_k r_k \qquad \leftrightsquigarrow \qquad \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{N}^k$$

Factorization homomorphism:

$$\begin{aligned} \pi : \mathbb{N}^k &\longrightarrow \langle r_1, \ldots, r_k \rangle \\ \mathbf{a} &\longmapsto a_1 r_1 + \cdots + a_k r_k \end{aligned}$$

## Definition

A *minimal presentation* $\rho$ of $S$ is a minimal subset $\rho \subset \ker \pi$ whose reflexive, symmetric, transitive, and translation closure equals $\ker \pi$.

$S = \langle 6, 9, 20 \rangle$: $\rho = \{((3,0,0),(0,2,0)),((7,2,0),(0,0,3))\} \subset \ker \pi$

$\pi^{-1}(60)$:

bettiexample.pdf

$$\begin{aligned} ((7,2,0),(4,4,0)) = {} & ((3,0,0),(0,2,0)) \\ & + ((4,2,0),(4,2,0)) \end{aligned}$$

# Kernel congruences and minimal presentations

Let $S = \langle r_1, \ldots, r_k \rangle$.

$$n = a_1 r_1 + \cdots + a_k r_k \qquad \leftrightsquigarrow \qquad \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{N}^k$$

Factorization homomorphism:

$$\begin{array}{rcl} \pi : \mathbb{N}^k & \longrightarrow & \langle r_1, \ldots, r_k \rangle \\ \mathbf{a} & \longmapsto & a_1 r_1 + \cdots + a_k r_k \end{array}$$

## Definition

A *minimal presentation* $\rho$ of $S$ is a minimal subset $\rho \subset \ker \pi$ whose reflexive, symmetric, transitive, and translation closure equals $\ker \pi$.

$S = \langle 6, 9, 20 \rangle$: $\rho = \{((3,0,0),(0,2,0)),((7,2,0),(0,0,3))\} \subset \ker \pi$

$\pi^{-1}(60)$:

bettiexample.pdf

$$((7,2,0),(4,4,0)) = ((3,0,0),(0,2,0)) \\ + ((4,2,0),(4,2,0))$$

$\mathrm{Cong}(\rho) = \ker \pi$ when the graph on $\pi^{-1}(n)$ is connected for all $n \in S$.

# The shifting map

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \qquad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

# The shifting map

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \qquad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

# The shifting map

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \qquad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_k, \mathbf{a}' + \ell \mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_0, \mathbf{a}' + \ell \mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = \big||\mathbf{a}| - |\mathbf{a}'|\big|$.

# The shifting map

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \qquad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell\mathbf{e}_k, \mathbf{a}' + \ell\mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell\mathbf{e}_0, \mathbf{a}' + \ell\mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = \big||\mathbf{a}| - |\mathbf{a}'|\big|$.

Sneak peek for $M_n = \langle n, n+6, n+9, n+20 \rangle$ and $n \gg 0$:

# The shifting map

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \qquad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_k, \mathbf{a}' + \ell \mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_0, \mathbf{a}' + \ell \mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = \big||\mathbf{a}| - |\mathbf{a}'|\big|$.

Sneak peek for $M_n = \langle n, n+6, n+9, n+20 \rangle$ and $n \gg 0$:

$M_{450}$:

$$\left\{ \begin{array}{l} ((\ \ 0,0,8,0),(3,2,0,\ \ 3)),((\ \ 0,1,6,0),(4,0,0,\ \ 3)),((\ \ 0,3,0,0),(1,0,2,\ \ 0)), \\ ((20,5,0,0),(0,0,0,24)),((25,1,0,0),(0,0,4,21)),((26,0,0,0),(0,2,2,21)) \end{array} \right\}$$

# The shifting map

$$M_n = \langle n, n+r_1, \ldots, n+r_k \rangle, \qquad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_k, \mathbf{a}' + \ell \mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_0, \mathbf{a}' + \ell \mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = \big||\mathbf{a}| - |\mathbf{a}'|\big|$.

Sneak peek for $M_n = \langle n, n+6, n+9, n+20 \rangle$ and $n \gg 0$:

$M_{450}$:

$$\left\{ \begin{array}{l} ((\ \ 0,0,8,0),(3,2,0,\ \ 3)),((\ \ 0,1,6,0),(4,0,0,\ \ 3)),((\ \ 0,3,0,0),(1,0,2,\ \ 0)), \\ ((20,5,0,0),(0,0,0,24)),((25,1,0,0),(0,0,4,21)),((26,0,0,0),(0,2,2,21)) \end{array} \right\}$$

$M_{470}$:

$$\left\{ \begin{array}{l} ((\ \ 0,0,8,0),(3,2,0,\ \ 3)),((\ \ 0,1,6,0),(4,0,0,\ \ 3)),((\ \ 0,3,0,0),(1,0,2,\ \ 0)), \\ ((21,5,0,0),(0,0,0,25)),((26,1,0,0),(0,0,4,22)),((27,0,0,0),(0,2,2,22)) \end{array} \right\}$$

# The shifting map

$$M_n = \langle n, n+r_1, \ldots, n+r_k \rangle, \qquad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_k, \mathbf{a}' + \ell \mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_0, \mathbf{a}' + \ell \mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = \big||\mathbf{a}| - |\mathbf{a}'|\big|$.

Sneak peek for $M_n = \langle n, n+6, n+9, n+20 \rangle$ and $n \gg 0$:

$M_{450}$:

$$\left\{ \begin{array}{l} ((\ 0,0,8,0),(3,2,0,\ 3)),((\ 0,1,6,0),(4,0,0,\ 3)),((\ 0,3,0,0),(1,0,2,\ 0)), \\ ((20,5,0,0),(0,0,0,24)),((25,1,0,0),(0,0,4,21)),((26,0,0,0),(0,2,2,21)) \end{array} \right\}$$

$M_{470}$:

$$\left\{ \begin{array}{l} ((\ 0,0,8,0),(3,2,0,\ 3)),((\ 0,1,6,0),(4,0,0,\ 3)),((\ 0,3,0,0),(1,0,2,\ 0)), \\ ((21,5,0,0),(0,0,0,25)),((26,1,0,0),(0,0,4,22)),((27,0,0,0),(0,2,2,22)) \end{array} \right\}$$

$M_{490}$:

$$\left\{ \begin{array}{l} ((\ 0,0,8,0),(3,2,0,\ 3)),((\ 0,1,6,0),(4,0,0,\ 3)),((\ 0,3,0,0),(1,0,2,\ 0)), \\ ((22,5,0,0),(0,0,0,26)),((27,1,0,0),(0,0,4,23)),((28,0,0,0),(0,2,2,23)) \end{array} \right\}$$

# The shifting map

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \qquad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_k, \mathbf{a}' + \ell \mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_0, \mathbf{a}' + \ell \mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = \big||\mathbf{a}| - |\mathbf{a}'|\big|$.

# The shifting map

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \qquad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_k, \mathbf{a}' + \ell \mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_0, \mathbf{a}' + \ell \mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = \big||\mathbf{a}| - |\mathbf{a}'|\big|$.

- $\Phi_n$ is well-defined.

# The shifting map

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \qquad \pi_n : \mathbb{N}^{k+1} \quad\longrightarrow\quad M_n$$

Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \quad\longmapsto\quad \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell\mathbf{e}_k, \mathbf{a}' + \ell\mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell\mathbf{e}_0, \mathbf{a}' + \ell\mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = \big||\mathbf{a}| - |\mathbf{a}'|\big|$.

- $\Phi_n$ is well-defined.

$$\begin{aligned} \pi_n(\mathbf{a}) = a_0 n + \textstyle\sum_{i=1}^k a_i(n + r_i) &= |\mathbf{a}|n + \textstyle\sum_{i=1}^k a_i r_i \\ \pi_{n+r_k}(\mathbf{a}) = \qquad\qquad\qquad\qquad &= |\mathbf{a}|n + |\mathbf{a}|r_k + \textstyle\sum_{i=1}^k a_i r_i \end{aligned}$$

# The shifting map

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \qquad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_k, \mathbf{a}' + \ell \mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_0, \mathbf{a}' + \ell \mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = \big||\mathbf{a}| - |\mathbf{a}'|\big|$.

- $\Phi_n$ is well-defined.

$$\begin{aligned} \pi_n(\mathbf{a}) = a_0 n + \textstyle\sum_{i=1}^k a_i(n + r_i) &= |\mathbf{a}|n + \textstyle\sum_{i=1}^k a_i r_i \\ \pi_{n+r_k}(\mathbf{a}) = \qquad\qquad &= |\mathbf{a}|n + |\mathbf{a}|r_k + \textstyle\sum_{i=1}^k a_i r_i \end{aligned}$$

- $\Phi_n$ preserves reflexive and symmetric closure.

# The shifting map

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \qquad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_k, \mathbf{a}' + \ell \mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_0, \mathbf{a}' + \ell \mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = \big||\mathbf{a}| - |\mathbf{a}'|\big|$.

- $\Phi_n$ is well-defined.

$$\begin{aligned} \pi_n(\mathbf{a}) &= a_0 n + \textstyle\sum_{i=1}^k a_i(n + r_i) = |\mathbf{a}|n + \sum_{i=1}^k a_i r_i \\ \pi_{n+r_k}(\mathbf{a}) &= \qquad\qquad\qquad\qquad\quad\; = |\mathbf{a}|n + |\mathbf{a}|r_k + \textstyle\sum_{i=1}^k a_i r_i \end{aligned}$$

- $\Phi_n$ preserves reflexive and symmetric closure.
- $\Phi_n$ preserves translation closure.

# The shifting map

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \qquad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_k, \mathbf{a}' + \ell \mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_0, \mathbf{a}' + \ell \mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = \big| |\mathbf{a}| - |\mathbf{a}'| \big|$.

- $\Phi_n$ is well-defined.

$$\begin{aligned} \pi_n(\mathbf{a}) = a_0 n + \textstyle\sum_{i=1}^k a_i(n + r_i) &= |\mathbf{a}|n + \textstyle\sum_{i=1}^k a_i r_i \\ \pi_{n+r_k}(\mathbf{a}) = \qquad &= |\mathbf{a}|n + |\mathbf{a}|r_k + \textstyle\sum_{i=1}^k a_i r_i \end{aligned}$$

- $\Phi_n$ preserves reflexive and symmetric closure.
- $\Phi_n$ preserves translation closure.

$$\Phi_n((\mathbf{a}, \mathbf{a}') + (\mathbf{b}, \mathbf{b})) = \Phi_n(\mathbf{a}, \mathbf{a}') + (\mathbf{b}, \mathbf{b})$$

# The shifting map

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \qquad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_k, \mathbf{a}' + \ell \mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_0, \mathbf{a}' + \ell \mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = \big||\mathbf{a}| - |\mathbf{a}'|\big|$.

- $\Phi_n$ is well-defined.

$$\begin{aligned} \pi_n(\mathbf{a}) = a_0 n + \textstyle\sum_{i=1}^k a_i(n + r_i) &= |\mathbf{a}|n + \textstyle\sum_{i=1}^k a_i r_i \\ \pi_{n+r_k}(\mathbf{a}) = \qquad\qquad\qquad\qquad &= |\mathbf{a}|n + |\mathbf{a}|r_k + \textstyle\sum_{i=1}^k a_i r_i \end{aligned}$$

- $\Phi_n$ preserves reflexive and symmetric closure.
- $\Phi_n$ preserves translation closure.

$$\Phi_n((\mathbf{a}, \mathbf{a}') + (\mathbf{b}, \mathbf{b})) = \Phi_n(\mathbf{a}, \mathbf{a}') + (\mathbf{b}, \mathbf{b})$$

- Only missing link: transitivity.

# Monotone chains

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \qquad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_k, \mathbf{a}' + \ell \mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_0, \mathbf{a}' + \ell \mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = \big||\mathbf{a}| - |\mathbf{a}'|\big|$.

# Monotone chains

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \qquad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_k, \mathbf{a}' + \ell \mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_0, \mathbf{a}' + \ell \mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = \big| |\mathbf{a}| - |\mathbf{a}'| \big|$.

Fix $(\mathbf{a}, \mathbf{b}), (\mathbf{b}, \mathbf{c}) \in \ker \pi_n$ with

$$|\mathbf{a}| < |\mathbf{b}| < |\mathbf{c}|.$$

# Monotone chains

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \qquad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_k, \mathbf{a}' + \ell \mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_0, \mathbf{a}' + \ell \mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = \big||\mathbf{a}| - |\mathbf{a}'|\big|$.

Fix $(\mathbf{a}, \mathbf{b}), (\mathbf{b}, \mathbf{c}) \in \ker \pi_n$ with

$$|\mathbf{a}| < |\mathbf{b}| < |\mathbf{c}|.$$

monotonechain01.pdf

# Monotone chains

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \qquad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_k, \mathbf{a}' + \ell \mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_0, \mathbf{a}' + \ell \mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = \big| |\mathbf{a}| - |\mathbf{a}'| \big|$.

Fix $(\mathbf{a}, \mathbf{b}), (\mathbf{b}, \mathbf{c}) \in \ker \pi_n$ with

$$|\mathbf{a}| < |\mathbf{b}| < |\mathbf{c}|.$$

monotonechain02.pdf

# Monotone chains

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \qquad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_k, \mathbf{a}' + \ell \mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_0, \mathbf{a}' + \ell \mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = \big| |\mathbf{a}| - |\mathbf{a}'| \big|$.

Fix $(\mathbf{a}, \mathbf{b}), (\mathbf{b}, \mathbf{c}) \in \ker \pi_n$ with

$$|\mathbf{a}| < |\mathbf{b}| < |\mathbf{c}|.$$

monotonechain03.pdf

# Monotone chains

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \qquad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_k, \mathbf{a}' + \ell \mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_0, \mathbf{a}' + \ell \mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = \big||\mathbf{a}| - |\mathbf{a}'|\big|$.

Fix $(\mathbf{a}, \mathbf{b}), (\mathbf{b}, \mathbf{c}) \in \ker \pi_n$ with

$$|\mathbf{a}| < |\mathbf{b}| < |\mathbf{c}|.$$

monotonechain04.pdf

# Monotone chains

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \qquad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell\mathbf{e}_k, \mathbf{a}' + \ell\mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell\mathbf{e}_0, \mathbf{a}' + \ell\mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = \big||\mathbf{a}| - |\mathbf{a}'|\big|$.

Fix $(\mathbf{a}, \mathbf{b}), (\mathbf{b}, \mathbf{c}) \in \ker \pi_n$ with

$$|\mathbf{a}| < |\mathbf{b}| < |\mathbf{c}|.$$

monotonechain05.pdf

# Monotone chains

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \qquad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell\mathbf{e}_k, \mathbf{a}' + \ell\mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell\mathbf{e}_0, \mathbf{a}' + \ell\mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = \big||\mathbf{a}| - |\mathbf{a}'|\big|$.

Fix $(\mathbf{a}, \mathbf{b}), (\mathbf{b}, \mathbf{c}) \in \ker \pi_n$ with

$$|\mathbf{a}| < |\mathbf{b}| < |\mathbf{c}|.$$

monotonechain06.pdf

# Monotone chains

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \qquad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell\mathbf{e}_k, \mathbf{a}' + \ell\mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell\mathbf{e}_0, \mathbf{a}' + \ell\mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = \big||\mathbf{a}| - |\mathbf{a}'|\big|$.

Fix $(\mathbf{a}, \mathbf{b}), (\mathbf{b}, \mathbf{c}) \in \ker \pi_n$ with

$$|\mathbf{a}| < |\mathbf{b}| < |\mathbf{c}|.$$

monotonechain07.pdf

# Monotone chains

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \qquad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_k, \mathbf{a}' + \ell \mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_0, \mathbf{a}' + \ell \mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = \big| |\mathbf{a}| - |\mathbf{a}'| \big|$.

Fix $(\mathbf{a}, \mathbf{b}), (\mathbf{b}, \mathbf{c}) \in \ker \pi_n$ with

$$|\mathbf{a}| < |\mathbf{b}| < |\mathbf{c}|.$$

monotonechain08.pdf

# Monotone chains

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \qquad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell\mathbf{e}_k, \mathbf{a}' + \ell\mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell\mathbf{e}_0, \mathbf{a}' + \ell\mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = \big||\mathbf{a}| - |\mathbf{a}'|\big|$.

Fix $(\mathbf{a}, \mathbf{b}), (\mathbf{b}, \mathbf{c}) \in \ker \pi_n$ with

$$|\mathbf{a}| < |\mathbf{b}| < |\mathbf{c}|.$$

`monotonechain09.pdf`

# Monotone chains

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \qquad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

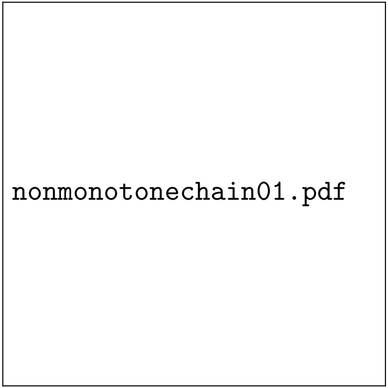
Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_k, \mathbf{a}' + \ell \mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_0, \mathbf{a}' + \ell \mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = \big| |\mathbf{a}| - |\mathbf{a}'| \big|$.

Fix $(\mathbf{a}, \mathbf{b}), (\mathbf{b}, \mathbf{c}) \in \ker \pi_n$ with

$$|\mathbf{a}| < |\mathbf{c}| < |\mathbf{b}|.$$



# Monotone chains

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \qquad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_k, \mathbf{a}' + \ell \mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_0, \mathbf{a}' + \ell \mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = \big||\mathbf{a}| - |\mathbf{a}'|\big|$.

Fix $(\mathbf{a}, \mathbf{b}), (\mathbf{b}, \mathbf{c}) \in \ker \pi_n$ with

$$|\mathbf{a}| < |\mathbf{c}| < |\mathbf{b}|.$$

nonmonotonechain02.pdf

# Monotone chains

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \qquad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_k, \mathbf{a}' + \ell \mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_0, \mathbf{a}' + \ell \mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = \big||\mathbf{a}| - |\mathbf{a}'|\big|$.

Fix $(\mathbf{a}, \mathbf{b}), (\mathbf{b}, \mathbf{c}) \in \ker \pi_n$ with

$$|\mathbf{a}| < |\mathbf{c}| < |\mathbf{b}|.$$

nonmonotonechain03.pdf

# Monotone chains

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \qquad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_k, \mathbf{a}' + \ell \mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_0, \mathbf{a}' + \ell \mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = \big||\mathbf{a}| - |\mathbf{a}'|\big|$.

Fix $(\mathbf{a}, \mathbf{b}), (\mathbf{b}, \mathbf{c}) \in \ker \pi_n$ with

$$|\mathbf{a}| < |\mathbf{c}| < |\mathbf{b}|.$$

nonmonotonechain04.pdf

# Monotone chains

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \qquad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_k, \mathbf{a}' + \ell \mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_0, \mathbf{a}' + \ell \mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

where $\ell = \big||\mathbf{a}| - |\mathbf{a}'|\big|$.

Fix $(\mathbf{a}, \mathbf{b}), (\mathbf{b}, \mathbf{c}) \in \ker \pi_n$ with

$$|\mathbf{a}| < |\mathbf{c}| < |\mathbf{b}|.$$

Need: *monotone* chains are sufficient for transitive closure.

`nonmonotonechain04.pdf`

# The main result

**Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)**

*For any $n > r_k^2$, the image $\Phi_n(\ker \pi_n)$ generates $\ker \pi_{n+r_k}$ as a congruence.*

# The main result

**Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)**

*For any $n > r_k^2$, the image $\Phi_n(\ker \pi_n)$ generates $\ker \pi_{n+r_k}$ as a congruence.*

$$\{\textit{minimal presentations of } M_n\} \quad \leftrightsquigarrow \quad \{\textit{minimal presentations of } M_{n+r_k}\}$$
$$\rho \subset \ker \pi_n \quad \longmapsto \quad \Phi_n(\rho) \subset \ker \pi_{n+r_k}$$

# The main result

## Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)

*For any* $n > r_k^2$*, the image* $\Phi_n(\ker \pi_n)$ *generates* $\ker \pi_{n+r_k}$ *as a congruence.*

$$\begin{array}{ccc} \{\textit{minimal presentations of } M_n\} & \leftrightsquigarrow & \{\textit{minimal presentations of } M_{n+r_k}\} \\ \rho \subset \ker \pi_n & \longmapsto & \Phi_n(\rho) \subset \ker \pi_{n+r_k} \end{array}$$

## Key Lemma

If $n > r_k^2$ and $(\mathbf{a}, \mathbf{a}') \in \rho$ with $|\mathbf{a}| > |\mathbf{a}'|$, then $z_0 > 0$ and $z_k' > 0$.

# The main result

**Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)**

*For any $n > r_k^2$, the image $\Phi_n(\ker \pi_n)$ generates $\ker \pi_{n+r_k}$ as a congruence.*

$$\begin{array}{ccc} \{\textit{minimal presentations of } M_n\} & \leftrightsquigarrow & \{\textit{minimal presentations of } M_{n+r_k}\} \\ \rho \subset \ker \pi_n & \longmapsto & \Phi_n(\rho) \subset \ker \pi_{n+r_k} \end{array}$$

**Key Lemma**

If $n > r_k^2$ and $(\mathbf{a}, \mathbf{a}') \in \rho$ with $|\mathbf{a}| > |\mathbf{a}'|$, then $z_0 > 0$ and $z_k' > 0$.

The above lemma ensures:

# The main result

**Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)**

*For any $n > r_k^2$, the image $\Phi_n(\ker \pi_n)$ generates $\ker \pi_{n+r_k}$ as a congruence.*

$$\begin{array}{rcl} \{\textit{minimal presentations of } M_n\} & \leftrightsquigarrow & \{\textit{minimal presentations of } M_{n+r_k}\} \\ \rho \subset \ker \pi_n & \longmapsto & \Phi_n(\rho) \subset \ker \pi_{n+r_k} \end{array}$$

**Key Lemma**

If $n > r_k^2$ and $(\mathbf{a}, \mathbf{a}') \in \rho$ with $|\mathbf{a}| > |\mathbf{a}'|$, then $z_0 > 0$ and $z_k' > 0$.

The above lemma ensures:

- Each $(\mathbf{a}, \mathbf{a}') \in \rho$ lies in the image of $\Phi_n$.

# The main result

## Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)

*For any $n > r_k^2$, the image $\Phi_n(\ker \pi_n)$ generates $\ker \pi_{n+r_k}$ as a congruence.*

$$\{\textit{minimal presentations of } M_n\} \quad \leftrightsquigarrow \quad \{\textit{minimal presentations of } M_{n+r_k}\}$$
$$\rho \subset \ker \pi_n \quad \longmapsto \quad \Phi_n(\rho) \subset \ker \pi_{n+r_k}$$

## Key Lemma

If $n > r_k^2$ and $(\mathbf{a}, \mathbf{a}') \in \rho$ with $|\mathbf{a}| > |\mathbf{a}'|$, then $z_0 > 0$ and $z_k' > 0$.

The above lemma ensures:

- Each $(\mathbf{a}, \mathbf{a}') \in \rho$ lies in the image of $\Phi_n$.
- Any two factorizations are connected by a monotone $\rho$-chain.

# The main result

**Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)**

*For any $n > r_k^2$, the image $\Phi_n(\ker \pi_n)$ generates $\ker \pi_{n+r_k}$ as a congruence.*

$$\begin{array}{ccc} \{\textit{minimal presentations of } M_n\} & \leftrightsquigarrow & \{\textit{minimal presentations of } M_{n+r_k}\} \\ \rho \subset \ker \pi_n & \longmapsto & \Phi_n(\rho) \subset \ker \pi_{n+r_k} \end{array}$$

**Key Lemma**

If $n > r_k^2$ and $(\mathbf{a}, \mathbf{a}') \in \rho$ with $|\mathbf{a}| > |\mathbf{a}'|$, then $z_0 > 0$ and $z_k' > 0$.

The above lemma ensures:

- Each $(\mathbf{a}, \mathbf{a}') \in \rho$ lies in the image of $\Phi_n$.
- Any two factorizations are connected by a monotone $\rho$-chain.

# The main result

## Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)

*For any $n > r_k^2$, the image $\Phi_n(\ker \pi_n)$ generates $\ker \pi_{n+r_k}$ as a congruence.*

$$\begin{array}{ccc} \{\textit{minimal presentations of } M_n\} & \leftrightsquigarrow & \{\textit{minimal presentations of } M_{n+r_k}\} \\ \rho \subset \ker \pi_n & \longmapsto & \Phi_n(\rho) \subset \ker \pi_{n+r_k} \end{array}$$

## Key Lemma

If $n > r_k^2$ and $(\mathbf{a}, \mathbf{a}') \in \rho$ with $|\mathbf{a}| > |\mathbf{a}'|$, then $z_0 > 0$ and $z_k' > 0$.

The above lemma ensures:

- Each $(\mathbf{a}, \mathbf{a}') \in \rho$ lies in the image of $\Phi_n$.
- Any two factorizations are connected by a monotone $\rho$-chain.

# The main result

**Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)**

*For any $n > r_k^2$, the image $\Phi_n(\ker \pi_n)$ generates $\ker \pi_{n+r_k}$ as a congruence.*

$$\begin{array}{rcl} \{\textit{minimal presentations of } M_n\} & \leftrightsquigarrow & \{\textit{minimal presentations of } M_{n+r_k}\} \\ \rho \subset \ker \pi_n & \longmapsto & \Phi_n(\rho) \subset \ker \pi_{n+r_k} \end{array}$$

**Key Lemma**

If $n > r_k^2$ and $(\mathbf{a}, \mathbf{a}') \in \rho$ with $|\mathbf{a}| > |\mathbf{a}'|$, then $z_0 > 0$ and $z_k' > 0$.

The above lemma ensures:

- Each $(\mathbf{a}, \mathbf{a}') \in \rho$ lies in the image of $\Phi_n$.
- Any two factorizations are connected by a monotone $\rho$-chain.

# The main result

## Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)

*For any $n > r_k^2$, the image $\Phi_n(\ker \pi_n)$ generates $\ker \pi_{n+r_k}$ as a congruence.*

$$\begin{array}{ccc} \{\textit{minimal presentations of } M_n\} & \leftrightsquigarrow & \{\textit{minimal presentations of } M_{n+r_k}\} \\ \rho \subset \ker \pi_n & \longmapsto & \Phi_n(\rho) \subset \ker \pi_{n+r_k} \end{array}$$

## Key Lemma

If $n > r_k^2$ and $(\mathbf{a}, \mathbf{a}') \in \rho$ with $|\mathbf{a}| > |\mathbf{a}'|$, then $z_0 > 0$ and $z_k' > 0$.

The above lemma ensures:

- Each $(\mathbf{a}, \mathbf{a}') \in \rho$ lies in the image of $\Phi_n$.
- Any two factorizations are connected by a monotone $\rho$-chain.

# The main result

**Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)**

*For any $n > r_k^2$, the image $\Phi_n(\ker \pi_n)$ generates $\ker \pi_{n+r_k}$ as a congruence.*

$$\begin{array}{ccc} \{\textit{minimal presentations of } M_n\} & \leftrightsquigarrow & \{\textit{minimal presentations of } M_{n+r_k}\} \\ \rho \subset \ker \pi_n & \longmapsto & \Phi_n(\rho) \subset \ker \pi_{n+r_k} \end{array}$$

**Key Lemma**

If $n > r_k^2$ and $(\mathbf{a}, \mathbf{a}') \in \rho$ with $|\mathbf{a}| > |\mathbf{a}'|$, then $z_0 > 0$ and $z_k' > 0$.

The above lemma ensures:

- Each $(\mathbf{a}, \mathbf{a}') \in \rho$ lies in the image of $\Phi_n$.
- Any two factorizations are connected by a monotone $\rho$-chain.

# The main result

**Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)**

*For any $n > r_k^2$, the image $\Phi_n(\ker \pi_n)$ generates $\ker \pi_{n+r_k}$ as a congruence.*

$$\begin{array}{ccc} \{\textit{minimal presentations of } M_n\} & \leftrightsquigarrow & \{\textit{minimal presentations of } M_{n+r_k}\} \\ \rho \subset \ker \pi_n & \longmapsto & \Phi_n(\rho) \subset \ker \pi_{n+r_k} \end{array}$$

**Key Lemma**

If $n > r_k^2$ and $(\mathbf{a}, \mathbf{a}') \in \rho$ with $|\mathbf{a}| > |\mathbf{a}'|$, then $z_0 > 0$ and $z_k' > 0$.

The above lemma ensures:

- Each $(\mathbf{a}, \mathbf{a}') \in \rho$ lies in the image of $\Phi_n$.
- Any two factorizations are connected by a monotone $\rho$-chain.

# The main result

## Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)

*For any $n > r_k^2$, the image $\Phi_n(\ker \pi_n)$ generates $\ker \pi_{n+r_k}$ as a congruence.*

$$\begin{array}{ccc} \{\textit{minimal presentations of } M_n\} & \leftrightsquigarrow & \{\textit{minimal presentations of } M_{n+r_k}\} \\ \rho \subset \ker \pi_n & \longmapsto & \Phi_n(\rho) \subset \ker \pi_{n+r_k} \end{array}$$

## Key Lemma

If $n > r_k^2$ and $(\mathbf{a}, \mathbf{a}') \in \rho$ with $|\mathbf{a}| > |\mathbf{a}'|$, then $z_0 > 0$ and $z_k' > 0$.

The above lemma ensures:

- Each $(\mathbf{a}, \mathbf{a}') \in \rho$ lies in the image of $\Phi_n$.
- Any two factorizations are connected by a monotone $\rho$-chain.

# The main result

**Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)**

*For any $n > r_k^2$, the image $\Phi_n(\ker \pi_n)$ generates $\ker \pi_{n+r_k}$ as a congruence.*

$$\begin{array}{rcl} \{\textit{minimal presentations of } M_n\} & \leftrightsquigarrow & \{\textit{minimal presentations of } M_{n+r_k}\} \\ \rho \subset \ker \pi_n & \longmapsto & \Phi_n(\rho) \subset \ker \pi_{n+r_k} \end{array}$$

**Key Lemma**

If $n > r_k^2$ and $(\mathbf{a}, \mathbf{a}') \in \rho$ with $|\mathbf{a}| > |\mathbf{a}'|$, then $z_0 > 0$ and $z_k' > 0$.

The above lemma ensures:

- Each $(\mathbf{a}, \mathbf{a}') \in \rho$ lies in the image of $\Phi_n$.
- Any two factorizations are connected by a monotone $\rho$-chain.

# The main result

## Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)

*For any $n > r_k^2$, the image $\Phi_n(\ker \pi_n)$ generates $\ker \pi_{n+r_k}$ as a congruence.*

$$\begin{array}{ccc} \{\textit{minimal presentations of } M_n\} & \leftrightsquigarrow & \{\textit{minimal presentations of } M_{n+r_k}\} \\ \rho \subset \ker \pi_n & \longmapsto & \Phi_n(\rho) \subset \ker \pi_{n+r_k} \end{array}$$

## Key Lemma

If $n > r_k^2$ and $(\mathbf{a}, \mathbf{a}') \in \rho$ with $|\mathbf{a}| > |\mathbf{a}'|$, then $z_0 > 0$ and $z_k' > 0$.

The above lemma ensures:

- Each $(\mathbf{a}, \mathbf{a}') \in \rho$ lies in the image of $\Phi_n$.
- Any two factorizations are connected by a monotone $\rho$-chain.

# The main result

**Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)**

*For any $n > r_k^2$, the image $\Phi_n(\ker \pi_n)$ generates $\ker \pi_{n+r_k}$ as a congruence.*

$$\begin{array}{ccc} \{\textit{minimal presentations of } M_n\} & \leftrightsquigarrow & \{\textit{minimal presentations of } M_{n+r_k}\} \\ \rho \subset \ker \pi_n & \longmapsto & \Phi_n(\rho) \subset \ker \pi_{n+r_k} \end{array}$$

# The main result

**Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)**

*For any $n > r_k^2$, the image $\Phi_n(\ker \pi_n)$ generates $\ker \pi_{n+r_k}$ as a congruence.*

$$\begin{array}{ccc} \{\textit{minimal presentations of } M_n\} & \leftrightsquigarrow & \{\textit{minimal presentations of } M_{n+r_k}\} \\ \rho \subset \ker \pi_n & \longmapsto & \Phi_n(\rho) \subset \ker \pi_{n+r_k} \end{array}$$

Consequences:

# The main result

**Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)**

*For any $n > r_k^2$, the image $\Phi_n(\ker \pi_n)$ generates $\ker \pi_{n+r_k}$ as a congruence.*

$$\{\textit{minimal presentations of } M_n\} \quad \leftrightsquigarrow \quad \{\textit{minimal presentations of } M_{n+r_k}\}$$
$$\rho \subset \ker \pi_n \quad \longmapsto \quad \Phi_n(\rho) \subset \ker \pi_{n+r_k}$$

Consequences:

- The Betti numbers $n \mapsto \beta_0(M_n)$ are eventually $r_k$-periodic:

  Graded degrees for $\beta_0(M_n)$ are $\pi_n(\mathbf{a})$ for each $(\mathbf{a}, \mathbf{a}') \in \rho$

# The main result

**Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)**

*For any $n > r_k^2$, the image $\Phi_n(\ker \pi_n)$ generates $\ker \pi_{n+r_k}$ as a congruence.*

$$\begin{array}{ccc} \{\textit{minimal presentations of } M_n\} & \leftrightsquigarrow & \{\textit{minimal presentations of } M_{n+r_k}\} \\ \rho \subset \ker \pi_n & \longmapsto & \Phi_n(\rho) \subset \ker \pi_{n+r_k} \end{array}$$

Consequences:

- The Betti numbers $n \mapsto \beta_0(M_n)$ are eventually $r_k$-periodic:

  Graded degrees for $\beta_0(M_n)$ are $\pi_n(\mathbf{a})$ for each $(\mathbf{a}, \mathbf{a}') \in \rho$

- The function $n \mapsto \Delta(M_n)$ is eventually singleton:

  $\Delta(M_n) = \{d\}$ when $||\mathbf{a}| - |\mathbf{a}'|| \in \{0, d\}$ for all $(\mathbf{a}, \mathbf{a}') \in \rho$

# The main result

**Theorem (Conaway–Gotti–Horton–O.–Pelayo–Williams–Wissman)**

*For any $n > r_k^2$, the image $\Phi_n(\ker \pi_n)$ generates $\ker \pi_{n+r_k}$ as a congruence.*

$$\begin{array}{rcl} \{\textit{minimal presentations of } M_n\} & \leftrightsquigarrow & \{\textit{minimal presentations of } M_{n+r_k}\} \\ \rho \subset \ker \pi_n & \longmapsto & \Phi_n(\rho) \subset \ker \pi_{n+r_k} \end{array}$$

Consequences:

- The Betti numbers $n \mapsto \beta_0(M_n)$ are eventually $r_k$-periodic:

  Graded degrees for $\beta_0(M_n)$ are $\pi_n(\mathbf{a})$ for each $(\mathbf{a}, \mathbf{a}') \in \rho$

- The function $n \mapsto \Delta(M_n)$ is eventually singleton:

  $\Delta(M_n) = \{d\}$ when $||\mathbf{a}| - |\mathbf{a}'|| \in \{0, d\}$ for all $(\mathbf{a}, \mathbf{a}') \in \rho$

- The function $n \mapsto \mathrm{c}(M_n)$ is eventually $r_k$-quasilinear:

  $\mathrm{c}(M_n)$ is determined by $\{$minimal presentations of $M_n\}$

# References


S. Chapman, N. Kaplan, T. Lemburg, A. Niles, and C. Zlogar (2014),
*Shifts of generators and delta sets of numerical monoids*,
Internat. J. Algebra Comput. 24 (2014), no. 5, 655–669.

T. Vu (2014),
*Periodicity of Betti numbers of monomial curves*,
Journal of Algebra 418 (2014) 66–90.

R. Conaway, F. Gotti, J. Horton, C. O'Neill, R. Pelayo, M. Williams, and B. Wissman (2016)
*Minimal presentations of shifted numerical monoids*.
in preparation.

M. Delgado, P. García-Sánchez, and J. Morais,
*GAP numerical semigroups package*
http://www.gap-system.org/Packages/numericalsgps.html.

# References

S. Chapman, N. Kaplan, T. Lemburg, A. Niles, and C. Zlogar (2014),
*Shifts of generators and delta sets of numerical monoids*,
Internat. J. Algebra Comput. 24 (2014), no. 5, 655–669.

T. Vu (2014),
*Periodicity of Betti numbers of monomial curves*,
Journal of Algebra 418 (2014) 66–90.

R. Conaway, F. Gotti, J. Horton, C. O'Neill, R. Pelayo, M. Williams, and B. Wissman (2016)
*Minimal presentations of shifted numerical monoids*.
in preparation.

M. Delgado, P. García-Sánchez, and J. Morais,
*GAP numerical semigroups package*
http://www.gap-system.org/Packages/numericalsgps.html.

Thanks!