Outline

Ordinary Generating Functions

Exponential Generating Functions (EGF)

Rational Generating Functions

The Exponential Formula
Ordinary Generating Functions (OGF)

**Definition**

The formal series

\[ F(x) := \sum_{n \geq 0} f(n)x^n \in \mathbb{C}[[x]] \]

associated to the counting map \( f : \mathbb{N}_0 \to \mathbb{C} \) is called *ordinary generating function*. We also use the notation \([x^n]F(x) := f(n)\).
Sum and Product of OGF

We recall that $\mathbb{C}[[x]]$ is an integral domain (actually a PID). We can perform sums and multiplications of OGF according to the way we sum and multiply elements in $\mathbb{C}[[x]]$.

**Definition**

Let $F(x) = \sum_{n \geq 0} f(n)x^n$ and $G(n) = \sum_{n \geq 0} g(n)x^n$ be two OGF. Then their *sum* is the OGF

$$F(x) + G(x) := \sum_{n \geq 0} (f(n) + g(n))x^n,$$

and their *product*, also called their *convolution* is the OGF

$$F(x)G(x) := \sum_{n \geq 0} \left( \sum_{k=0}^{n} f(k)g(n-k) \right)x^n.$$

**Example:** The OGF of the Fibonacci sequence is $F(x) = \frac{1}{1-x-x^2}$. 

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The Inverse of an OGF

**Theorem**

An OGF \( F(x) \) is invertible (meaning that there exists an OGF \( G(x) \) such that \( F(x)G(x) = 1 \)) iff \( F(0) \neq 0 \).

**Proof:** If \( G(x) \) is the inverse of \( F(x) \) then \( F(0)G(0) = 1 \) and so \( F(0) \neq 0 \). If \( F(0) \neq 0 \) then \( G(0) = F(0)^{-1} \), and we can recurrently define the remaining coefficients of \( G(0) \) by using the multiplication formula for OGF.

**Example:** The OGF \( F(x) = 1 - x \) satisfies that \( F(0) = 1 \neq 0 \). Therefore, it is invertible. The inverse of \( F(x) \) is \( G(x) = \sum_{n \geq 0} x^n \).
Convergence of OGF

**Definition**

The *degree* of an OGF $F(x) = \sum_{n \geq 0} f(n)x^n$, denoted by $\deg F(x)$, is the smallest $n$ such that $f(n) \neq 0$. A sequence of OGF \( \{F_i(x)\}_{i \in \mathbb{N}} \) converges to the OGF $F(x)$ if

$$\lim_{i \to \infty} \deg(F(x) - F_i(x)) = \infty.$$ 

We say that the infinite sum $\sum_{i \geq 0} F_i$ converges to the OGF $F(x)$ if the sequence of partial sum converges to $F(x)$.

**Theorem**

*The infinite series $\sum_{i \geq 0} F_i(x)$ converges iff $\lim_{i \to \infty} \deg F_i(x) = \infty$.***

**Proof:** It follows directly from the definition of convergence. \qed
Composition of OGF

Definition

Let \( F(x) = \sum_{n \geq 0} f(n)x^n \) and \( G(n) = \sum_{n \geq 0} g(n)x^n \) be two OGF such that \( G(0) = 0 \). Then the composition of \( F(x) \) and \( G(x) \) is the OGF

\[
F(G(x)) := \sum_{n \geq 0} f(n)G(x)^n.
\]

Remarks:

- Notice that the condition \( G(0) = 0 \) guarantees that \( \sum_{n \geq 0} f(n)G(x)^n \) converges. This is because \( \text{deg } G(x)^n \geq n \text{deg } G(x) \), and so \( \lim_{i \to \infty} \text{deg} (f(i)G(x)^i) = \infty \).

- The expression \( e^{1+x} = \sum_{n \geq 0} (x + 1)^n / n! \) is not a valid OGF because the series does not converge in the sense we defined above.
A Few Popular OGFs

Theorem

The following are popular and useful OGFs:

1. \( \sum_{n \geq 0} x^n = \frac{1}{1-x} \),
2. \( \sum_{n \geq 0} (-1)^n x^n = \frac{1}{1+x} \),
3. \( \sum_{n \geq 0} x^{2n} = \frac{1}{1-x^2} \),
4. \( \sum_{n \geq 0} \binom{m}{n} x^n = (1 + x)^m \),
5. \( \sum_{n \geq 0} \binom{n+m}{n} x^n = \frac{1}{(1-x)^{m+1}} \),
6. \( \sum_{n \geq 0} \binom{n}{m} x^n = \frac{x^m}{(1-x)^{m+1}} \).

Sketch of Proof: Pending...
Definition

The formal series

\[ F(x) := \sum_{n \geq 0} f(n) \frac{x^n}{n!} \]

associated to the counting map \( f : \mathbb{N}_0 \to \mathbb{C} \) is called the exponential generating function (EGF) of \( f \). We also use the notation \([x^n/n!]F(x) := f(n)\).
Sum, Product, and Composition of EGFs

**Definition**

Let $F(x) = \sum_{n \geq 0} f(n)x^n/n!$ and $G(n) = \sum_{n \geq 0} g(n)x^n/n!$ be two EGF. Then their *sum* is the EGF

$$F(x) + G(x) := \sum_{n \geq 0} (f(n) + g(n))x^n/n!,$$

and their *product*, also called their *convolution* is the OGF

$$F(x)G(x) := \sum_{n \geq 0} \left( \sum_{k=0}^{n} \binom{n}{k} f(k)g(n-k) \right)x^n/n!.$$

If, in addition, $G(0) = 0$, we composition of the EGFs $F$ and $G$ is given by

$$F(G(x)) := \sum_{n \geq 0} f(n)G(x)^n/n!.$$
The formal derivative of a GF

**Definition**

- The derivative of the OGF $F(x) = \sum_{n \geq 0} f(n)x^n$ is $F'(x) := \sum_{n \geq 0} nf(n)x^{n-1}$.
- The derivative of the EGF $F(x) = \sum_{n \geq 0} f(n)x^n / n!$ is $F'(x) := \sum_{n \geq 0} f(n)x^{n-1}$.

**Theorem**

Let $F(x)$ and $G(x)$ be two OGF (EGF) then the following hold:

- $(F(x) + G(x))' = F'(x) + G'(x)$,
- $F(x)G(x) = F'(x)G(x) + F(x)G'(x)$,
- $(F(G(x)))' = F'(G(x))G'(x)$.
The formal derivative of a GF

**Theorem**

Let $F(x)$ and $G(x)$ be two OGFs such that $F(0) = 1$ and $G(0) = 0$. If $G'(x) = F'(x)/F(x)$ then $F(x) = \exp(G(x))$, where

$$\exp(G(x)) = \sum_{n \geq 0} \frac{G(x)^n}{n!}.$$  

**Sketch of Proof:** Pending...

**Example 1:** The EGF of the function $f : \mathbb{N} \rightarrow \mathbb{C}$ given by $f(0) = 1$ and $f(n + 1) = f(n) + nf(n - 1)$ if $n \geq 0$
General Newton Coefficients

Definition

- For $\lambda \in \mathbb{C}$ and $k \in \mathbb{N}_0$, set $\binom{\lambda}{k} := \lambda(\lambda - 1) \ldots (\lambda - k + 1)$.
- For an OGF $F(x)$ such that $F(0) = 0$, we define:

$$ (1 + F(x))^\lambda := \sum_{n \geq 0} \binom{\lambda}{n} F(x)^n. $$

Example 2: We want to find all $f : \mathbb{N}_0 \to \mathbb{R}$ satisfying

$$ \sum_{k=0}^{n} f(k)f(n - k) = 1. $$
A Practice Example

**Example:** Suppose that the function $f : \mathbb{N} \to \mathbb{C}$ has EGF $F(x) = e^x + x^2/2$.

- Find a recurrence formula for $f$.
- Find an explicit formula for $f$. 

Let \( f : \mathbb{N}_0 \rightarrow \mathbb{C} \) and
\[ Q(x) = 1 + c_1 x + \cdots + c_d x^d = \prod_{i=1}^{k} (1 - \alpha_i x)^{d_i}, \]
where \( c_1, \ldots, c_d \in \mathbb{C} \) \((c_d \neq 0)\). Then TFAE:

1. \( f(n+d) + c_1 f(n+d-1) + \cdots + c_d f(n) = 0 \) for every \( n \in \mathbb{N}_0; \)
2. \( F(x) = \sum_{n \geq 0} f(b)x^n = P(x)/Q(x) \), where \( \deg P(x) < d; \)
3. \( F(x) = \sum_{n \geq 0} f(n)x^n = \sum_{i=1}^{k} g_i(x)/(1 - \alpha_i x)^{d_i}; \)
4. \( f(n) = \sum_{i=1}^{k} p_i(n)\alpha_i^n \), where \( p_i(n) \) is a polynomial in \( n \) such that \( \deg p_i < d_i \) for each \( i \in \{1, \ldots, k\} \).

**Sketch of Proof:** For each \( i \in \{1, 2, 3, 4\} \), define the complex space
\[ V_i := \{ f : \mathbb{N}_0 \rightarrow \mathbb{C} \mid f \text{ satisfies (i)} \}. \]
Check that \( \dim V_i = d \) for each \( i \). Use this to check that \( V_1 = V_2 \)
and \( V_3 = V_4 \). Finally, show that \( V_3 \subseteq V_2. \)
Rational Generating Functions (continuation)

Definition

A generating function $F(x) = \sum_{n \geq 0} f(n)x^n$ satisfying any of the four conditions in the previous theorem is called a (proper) rational generating function.
Examples

Example 1: Let $f(n)$ be the number of paths with $n$ non-intersecting steps starting from $(0,0)$ with directions east, north, or west.

1. Find the generating function of $F$ of $f$.
2. Find a close formula for $f$.

Hint: Count the paths of length $n$ ending in EE, WW, and NE.

Example 2: Write $(\sqrt{2} + \sqrt{3})^{1980}$ in decimal form. What is the last digit before and the first digit after the decimal point?

Hint: Compute the generating function of $(\sqrt{2} + \sqrt{3})^{2n}$. 

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Putting Structures on Finite Sets

Theorem

Let $f_1, \ldots, f_n : \mathbb{N}_0 \to \mathbb{C}$, and denote by $E_{f_i}(x)$ the EGF of $f_i$. For every finite set $S$, let

$$h(|S|) = \sum_{(T_1, \ldots, T_n)} f_1(|T_1|) \cdots f_n(|T_n|),$$

where the sum runs over every ordered $n$-partition of $S$. Then the EGF $E_h(x)$ of $h$ satisfies that $E_h(x) = E_{f_1}(x) \cdots E_{f_n}(x)$.

Sketch of Proof: Suppose first that $n = 2$. If $|S| = s$, the fact that there are $\binom{s}{k}$ ordered partitions $(T_1, T_2)$ such that $|T_1| = k$ of $S$ implies that

$$h(s) = \sum_{k=0}^{s} \binom{s}{k} f_1(k)f_2(n - k).$$

Then $E_h(x) = E_{f_1}(x)E_{f_2}(x)$. Now extend to $n$ by induction. \qed
The Compositional Formula

**Theorem**

*Given* $f : \mathbb{N} \to \mathbb{C}$ and $g : \mathbb{N}_0 \to \mathbb{C}$ with $g(0) = 1$, and for every finite set $S$ let

$$h(0) = h([n]) = \sum_{\{T_1, \ldots, T_k\} \in \pi([n])} f(|T_1|) \ldots f(|T_k|)g(k)$$

*if* $|S| > 0$ and $h(0) = 1$. Then $E_h(x) = E_g(E_f(x))$.

**Sketch of Proof:** Defining, for every $k \in \{1, \ldots, n\}$

$$h_k(n) = \frac{1}{k!} \sum_{(T_1, \ldots, T_k)} f(|T_1|) \ldots f(|T_k|)g(k),$$

we have $h(n) = \sum h_k(n)$. By the previous theorem, $E_{h_k}(x) = g(k)/k!E_f(x)^k$. Hence

$$E_h(x) = \sum_{k \geq 1} g(k) \frac{E_f(x)^k}{k!} = E_g(E_f(x)).$$
The Compositional Formula: An Example

Example: In how many ways $h(n)$ we can form $n$ people into nonempty lines, and then arrange these lines in a circular order?

Explanation: Let $f(n)$ and $g(n)$ the number of ways to form $n$ people in a line and in a circle, respectively. Then $f(n) = n!$ and $g(n) = (n - 1)!$. So the EGFs of $f$ and $g$ are

$$E_f(x) = \sum_{n \geq 1} x^n = \frac{x}{1-x} \quad \text{and} \quad E_g(x) = \sum_{n \geq 1} \frac{x^n}{n} = \ln(1 - x)^{-1}.$$ 

Hence, using the previous theorem,

$$E_h(x) = E_g(E_f(x)) = \ln\left(\frac{1-x}{1-2x}\right) = \sum_{n \geq 1} (2^n - 1)(n-1)! \frac{x^n}{n!}.$$ 

Thus $h(n) = (2^n - 1)(n-1)!$. 

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References


