SOLUTIONS FOR QUIZ 3

Note: Most of the problems were taken from the textbook [1].

Problem 1. *Find the Maclaurin series of* $f(x) = \frac{1}{\sqrt{5-x}}$.

Solution: Notice that

$$\begin{aligned}
\frac{1}{\sqrt{5-x}} &= \frac{1}{\sqrt{5}}\left(1-\frac{x}{5}\right)^{-1/2} \\
&= \frac{1}{\sqrt{5}}\sum_{n=0}^{\infty}\binom{-1/2}{n}\left(-\frac{x}{5}\right)^{n} \\
&= \frac{1}{\sqrt{5}}\sum_{n=0}^{\infty}\frac{(-1)^n(-\frac{1}{2})(-\frac{1}{2}-1)\dots(-\frac{1}{2}-n+1)}{5^n n!}x^n \\
&= \frac{1}{\sqrt{5}}\sum_{n=0}^{\infty}\frac{(\frac{1}{2})(\frac{1}{2}+1)\dots(\frac{1}{2}+n-1)}{5^n n!}x^n \\
&= \frac{1}{\sqrt{5}}\sum_{n=0}^{\infty}\frac{1\cdot 3\cdot \ldots \cdot (2n-1)}{10^n n!}x^n.
\end{aligned}$$

□

Problem 2. *Evaluate* $\int_0^1 e^{-x^2}\,dx$ *correct to within an error of* 0.001. *[Hint: Use Taylor expansion, then use the Alternating Estimation Theorem.]*

Solution: The Taylor series for $f(x) = e^{-x^2}$ is

$$e^{-x^2} = \sum_{n=0}^{\infty}\frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{n!}.$$

Then

$$\begin{aligned}
\int_0^1 e^{-x^2}dx &= \int_0^1\left(\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{n!}\right)dx \\
&= \left(\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)n!}\right)\Bigg|_0^1 \\
&= \sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)n!} \\
&= 1-\frac{1}{3}+\frac{1}{10}-\frac{1}{42}+\frac{1}{216}-\frac{1}{1320}+\cdots.
\end{aligned}$$

After applying the Alternating Estimation Theorem to estimate the error, we see that the first term with absolute within the given value of 0.001 is the sixth term (i.e., $n = 5$)

$$\frac{1}{11 \cdot 5!} = \frac{1}{1320} < 0.001.$$

Hence,

$$\int_0^1 e^{-x^2} dx \approx 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216}.$$

is an approximation within the given error. □

Problem 3. *Solve the differential equation* $\frac{dy}{dx} = \frac{x^5+1}{x^3y^2+y^4x^3}$.

Solution: We can rewrite our initial equation as

$$\frac{dy}{dx} = \frac{x^5+1}{x^3(y^2+y^4)} = \left(\frac{x^5+1}{x^3}\right)\left(\frac{1}{y^2+y^4}\right)$$

and we obtain that

$$\int (y^2 + y^4) dy = \int \frac{x^5+1}{x^3} dx \tag{1}$$

Integrating equation (1), we have

$$\frac{1}{3}y^3 + \frac{1}{5}y^5 = \frac{1}{3}x^3 - \frac{1}{2x^2} + C.$$

□

Problem 4. *Suppose that a population develop according to the logistic equation* $dP/dt = 0.06P - 0.0006P^2$, *where* t *is measured in weeks. (a) What is the carrying capacity? (b) What is the value of* k*? (c) What are the equilibrium solutions. (d) If the initial population is* 50, *what is the population after* 10 *weeks?*

Solution: Rewriting the given logistic equation, we have

$$dP/dt = 0.06P(1 - 0.01P) = 0.06P\left(1 - \frac{P}{100}\right). \tag{2}$$

As a result, the carrying capacity is $M = 100$ (a) and $k = 0.06$ (b). For part (c), we need to find the values of P for which $dP/dt = 0$. Looking at (2), we see that $dP/dt = 0$ only when $P = 0$ or $P = 100$. Hence, the equilibrium solutions are $P = 0$ and $P = 100$. Lastly, for part (d), we need to find the solution to the logistic equation, namely, we need to find the values of A, k, and M and substitute them in the following equation

$$P(t) = \frac{M}{1 + Ae^{-kt}}.$$

In previous parts, we found the values of $M = 100$ and $k = 0.06$. To find A, we use the formula

$$A = \frac{M - P_0}{P_0} = \frac{100 - 50}{50} = 1.$$

Thus,

$$P(t) = \frac{100}{1 + e^{-0.06t}}.$$

Finally, the population after 10 weeks is $P(10) = 100/(1 + e^{-0.6})$.

□

Problem 5. *Solve the first-order linear differential equations:*

(1) $y' - y = e^x$
(2) $y' + 2xy = 1$.

Solution:

(1) We have $P(x) = -1$, $Q(x) = e^x$, and $I(x) = e^{\int -1dx} = e^{-x}$. Hence

$$y = (x + C)e^x.$$

(2) Multiplying by e^{x^2} the given equation, we obtain

$$e^{x^2} y' + e^{x^2} 2xy = e^{x^2}.$$

Moreover, integrating the previous equation, we have

$$e^{x^2} y = \int (e^{x^2} y)' = \int e^{x^2} dx$$

and, therefore,

$$y = \frac{\int e^{x^2} dx}{e^{x^2}}.$$

□

References

[1] J. Stewart: *Single Variable Calculus* 8th Edition, Cengage Learning, Boston 2015.