

SOLUTIONS FOR QUIZ 2

Note: Most of the problems were taken from the textbook [1].

Problem 1. Determine whether the sequence $a_n = \cos(\sqrt[n]{\pi^{207+n}})$ converges or diverges. If it is convergent, find the limit.

Solution: Since $\cos x$ and π^x are both continuous functions we have that

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \cos\left(\pi^{\frac{207}{n}+1}\right) = \cos\left(\pi^{\lim_{n \rightarrow \infty} \frac{207}{n}+1}\right) = \cos(\pi) = -1.$$

Hence $\{a_n\}$ is a convergent sequence. □

Problem 2. Determine whether the series

$$\sum_{n=1}^{\infty} \sqrt[3]{\frac{n^2}{n^2 + 20n + 9}}$$

is convergent or divergent. If it is convergent, find the sum.

Solution: As $\sqrt[3]{x}$ is a continuous function, it follows that

$$\lim_{n \rightarrow \infty} \sqrt[3]{\frac{n^2}{n^2 + 20n + 9}} = \sqrt[3]{\lim_{n \rightarrow \infty} \frac{1}{1 + 20/n + 9/n^2}} = \sqrt[3]{1} = 1.$$

Since the general term of the series does not converges to zero, the series diverges. □

Problem 3. Find the exact area of the surface of revolution obtained by rotating the curve $x = 1 + 2y^2$, $1 \leq y \leq 2$ about the x -axis.

Solution: From $x = 1 + 2y^2$, we find that $f(x) = y = \sqrt{\frac{x-1}{2}}$. Since $1 \leq y \leq 2$, one has that $3 \leq x \leq 9$. In addition,

$$1 + [f(x)']^2 = 1 + \left(\frac{1}{\sqrt{2}} \frac{1}{2\sqrt{x-1}}\right)^2 = 1 + \frac{1}{8(x-1)} = \frac{8x-7}{8(x-1)}.$$

Therefore the area of the surface of revolution is

$$\begin{aligned} 2\pi \int_3^9 \sqrt{\frac{x-1}{2}} \sqrt{\frac{8x-7}{8(x-1)}} dx &= \frac{\pi}{2} \int_3^9 (8x-7)^{1/2} dx \\ &= \frac{\pi}{16} \int_{17}^{65} u^{1/2} du = \frac{\pi}{24} (\sqrt{65^3} - \sqrt{17^3}). \end{aligned}$$

□

Problem 4. Use the Integral Test to determine whether the series $\sum_{n=1}^{\infty} n^2 \pi^{-n^3}$ is convergent or divergent.

Solution: The function $f(x) = x^2 \pi^{-x^3}$ is continuous and positive for all $x \geq 1$. Moreover,

$$f(x)' = \frac{\pi^{x^3} (2x - (3 \ln \pi) x^4)}{\pi^{2x^3}} < 0$$

for all $x \geq 1$. Then $f(x)$ is decreasing when $x \geq 1$. Taking $u = x^3$ below, we find that

$$\int_1^{\infty} f(x) dx = \frac{1}{3} \int_1^{\infty} \left(\frac{1}{\pi} \right)^u du = \frac{1}{3 \ln(1/\pi)} (-1/\pi) = \frac{1}{3\pi \ln \pi}.$$

Since the above improper integral is finite, the series converges by the Integral Test. $\square$

Problem 5. Determine whether the series

$$\sum_{n=1}^{\infty} \left(\frac{2}{(n+1)(n+3)} + \frac{3}{\pi^{n+1}} \right)$$

is convergent or divergent. If it is convergent, find the sum.

Solution: Notice that

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\frac{2}{(n+1)(n+3)} + \frac{3}{\pi^{n+1}} \right) &= \sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+3} \right) + \frac{3}{\pi^2} \sum_{n=0}^{\infty} \left(\frac{1}{\pi} \right)^n \\ &= \lim_{n \rightarrow \infty} \left(\frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3} \right) + \frac{3/\pi^2}{1 - 1/\pi} \\ &= \frac{5}{6} + \frac{3}{\pi^2 - \pi}. \end{aligned}$$

Hence the series converges to $5/6 + 3/(\pi^2 - \pi)$. $\square$

REFERENCES

- [1] J. Stewart: *Single Variable Calculus* 8th Edition, Cengage Learning, Boston 2015.