$\mathbb{R}^3$-Geometry, Inner Product and Cross Product

Problem 1 (Stewart, Exercise 12.1.22). *Find an equation of a sphere if one of its diameters has endpoints* $(5,4,3)$ *and* $(1,6,-9)$.

Problem 2 (Stewart, Exercise 12.1.45). *Find an equation of the set of all points equidistant from the points* $(-1,5,3)$ *and* $(6,2,-2)$. *Describe the set.*

Problem 3 (Stewart, Exercise 12.1.47). *Find the distance between the spheres* $x^2+y^2+z^2=4$ *and* $x^2+y^2+z^2=4x+4y+4z-11$.

Problem 4 (Stewart, Exercise 12.2.41). *Find the unit vectors that are parallel to the tangent line of the parabola* $y=x^2$ *at the point* $(2,4)$.

Problem 5 (Stewart, Exercise 12.2.48). *If* $r=\langle x,y\rangle$, $r_1=\langle x_1,y_1\rangle$, *and* $r_2=\langle x_2,y_2\rangle$, *describe the set of all points* (x,y) *such that* $|r-r_1|+|r-r_2|=k$, *where* $k>|r_1-r_2|$. *Find an equation for such a set of points in rectangular coordinates.*

Problem 6 (Stewart, Exercise 12.3.56). *Find the angle between a diagonal of a cube and a diagonal of one of its faces.*

Problem 7 (Stewart, Exercises 12.3.(61,62)). *For any two vectors* a *and* b *in* $\mathbb{R}^3$,

(1) show that $|a\cdot b|\leq |a|\,|b|$ *(Cauchy-Schwarz Inequality);*

(2) show that $|a+b|\leq |a|+|b|$ *(Triangular Inequality).*

Problem 8 (Stewart, Exercise 12.3.64). *Show that if* $u+v$ *and* $u-v$ *are orthogonal, then the vectors* u *and* v *must have the same lengths. Interpret such a result geometrically.*

Problem 9 (Stewart, Exercise 12.4.43). *If* $a\cdot b=\sqrt{3}$ *and* $a\times b=\langle 1,2,2\rangle$, *find the angle between* a *and* b.

Problem 10 (Stewart, Exercise 12.4.44). *(1) Find all vectors* v *such that* $\langle 1,2,1\rangle\times v=\langle 3,1,-5\rangle$.

(2) Explain why there is no vector v *such that* $\langle 1,2,1\rangle\times v=\langle 3,1,5\rangle$.

Problem 11 (Stewart, Exercise 12.4.47). *Show that* $|a\times b|^2=|a|^2|b|^2-(a\cdot b)^2$.

Problem 12 (Math W53, Quiz). *Let* $a=\langle 0,-2,0\rangle$ *and let* $b=\langle b_1,b_2,0\rangle$ *be a three-dimensional vector such that* $|b|=4$.

(1) Find all such b *for which* $a\cdot b$ *is maximized.*

(2) Find all such b *for which* $|a\times b|$ *is maximized.*

Problem 13 (Math W53, Quiz). *Find all* $t\in\mathbb{R}$ *such that the vectors* $a=\langle 1,2,3\rangle$, $b=\langle 3,5,7\rangle$, *and* $c=\langle t,1,1\rangle$ *are coplanar.*

Problem 14 (Math W53, Quiz)**.** *Calculate the area of the triangle whose vertices are* $A = (2, 2, 0)$*,* $B = (0, 2, 2)$*, and* $C = (2, 0, 2)$*.*

Problem 15 (Math W53, Quiz)**.** *Let* $a = \langle a_1, a_2 \rangle$ *and* $b = \langle b_1, b_2 \rangle$ *be two different vectors.*

(1) *Show that* $r = \langle x, y \rangle$ *satisfies the equation* $(r - a) \cdot (r - b) = 0$ *if and only if* (x, y) *lies on a certain circle* O*. Find the equation of* O*.*
(2) *Find the center and radius of* O *in terms of* a *and* b *(without involving the components of* a *and* b*).*

References

[1] J. Stewart: *Calculus* 8th Edition, Cengage Learning, Boston 2016.