MULTIVARIABLE CALCULUS

1. Line Integrals and Green's Theorem

Problem 1 (Stewart, Exercise 16.1.(25,26)). Find and sketch the gradient vector field of the following functions:

(1)
$$f(x,y) = \frac{1}{2}(x-y)^2$$
 (2) $f(x,y) = \frac{1}{2}(x^2-y^2).$

Problem 2 (Stewart, Exercise 16.2.(5,11,14)). Evaluate the following line integrals:

- (1) $\int_C (x^2y + \sin x) dy$, where C is the arc of the parabola $y = x^2$ from (0,0) to (π, π^2) ;
- (2) $\int_C x e^{yz} ds$, where C is the line segment from (0,0,0) to (1,2,3);
- (3) $\int_C y \, dx + z \, dyx \, dz$, where $C = (\sqrt{t}, t, t^2)$ for $1 \le t \le 4$.

Problem 3 (Stewart, Exercise 16.2.41). Find the work done by the force field

$$F(x, y, z) = \langle x - y^2, y - z^2, z - x^2 \rangle$$

on a particle that moves along the line segment from (0,0,1) to (2,1,0).

Problem 4 (Stewart, Exercise 16.2.(5,11,14)). Let C be a smooth curve given by a vector function r(t) for $a \le t \le b$, and let v be a constant vector

- (1) Show that $\int_C v \cdot dr = v \cdot (r(b) r(a)).$
- (2) Show that $\int_C r \cdot dr = \frac{1}{2} (|r(b)|^2 |r(a)|^2).$

Problem 5 (Stewart, Exercise 16.3.(14,18)). Find a function f such that $F = \nabla f$ and use it to evaluate $\int_C F \cdot dr$ along the given curve.

- (1) $F(x,y) = \langle (1+xy)e^{xy}, x^2e^{xy} \rangle$, where $C: r(t) = \langle \cos t, 2\sin t \rangle$ for $0 \le t \le \pi/2$.
- (2) $F(x, y, z) = \langle \sin y, x \cos y + \cos z, -y \sin z \rangle$, where $C : r(t) = \langle \sin t, t, 2t \rangle$ for $0 \le t \le \pi/2$.

Problem 6 (Stewart, Exercise 16.3.(19,20)). Show that the following line integrals are independent of paths evaluate them.

- (1) $\int_C 2xe^{-y} dx + (2y x^2e^{-y}) dy$, where C is any path from (1,0) to (2,1).
- (2) $\int_C \sin y \, dx + (x \cos y \sin y) \, dy$, where C is any path from (2,0) to $(1,\pi)$.

Problem 7 (Stewart, Exercise 16.3.(19,20)). Determine whether or not the given subsets of \mathbb{R}^2 are open, connected, and/or simply connected:

(1) $\{(x, y) \mid 0 < y < 3\};$

(2) {(x, y) | 1 < |x| < y}; (3) { $(x, y) | 1 \le x^2 + y^2 \le 4, y \ge 0$ }; (4) { $(x, y) | (x, y) \ne (2, 3)$ }.

Problem 8 (Cal Final, Summer 2018W). Suppose a parametrized curve

$$r(t) = \langle x(t), y(t), z(t) \rangle \quad for \quad 0 \le t \le 1$$

satisfies the equation xx'(t) + yy'(t) + zz'(t) = 0 for all t. If x(0) = y(0) = z(0) = 3and x(1) = y(1) = 2, find |z(1)|.

Problem 9 (Stewart, Exercises 16.4.(1,7)). Evaluate the following integral via Green's Theorem:

- (1) $\oint_C y^2 dx + x^2 y dy$, where C is the rectangle with vertices (0,0), (5,0), (5,4), and (0,4);
- (2) $\int_C (y + e^{\sqrt{x}}) dx + (2x + \cos y^2) dy$, where C is the boundary of the region enclosed by the parabolas $y = x^2$ and $x = y^2$.

Problem 10 (Stewart, Exercises 16.4.18). A particle starts at the origin, move along the x-axis to (5,0), then along the quarter-circle $x^2 + y^2 = 25$ where $x \ge 0$ and $y \ge 0$ to the point (0,5), and then down to the y-axis back to the origin. Find the work done on this particle by the force field $F = \langle \sin x, \sin y + xy^2 + \frac{1}{3}x^3 \rangle$.

Problem 11 (Stewart, Exercises 16.4.19). Let D be the region bounded by a positivelyoriented, piecewise-smooth, simple closed curve C.

- (1) Argue that $\operatorname{Area}(D) = \oint_C x \, dy = -\oint_C y \, dx = 1/2 \oint_C x \, dy y \, dx$.
- (2) Use the previous part to find the area under one arch of the cycloid $(t \sin t, 1 \cos t)$.

If the **density** of a solid occupying the region E is given by $\rho(x, y, z)$, then its mass can be computed by

$$m = \iiint_E \rho(x, y, z) \, dV$$

and its **center of mass** is $(\bar{x}, \bar{y}, \bar{z})$, where

$$\bar{x} = \iiint_E x\rho(x, y, z) \, dV; \quad \bar{y} = \iiint_E y\rho(x, y, z) \, dV; \quad \bar{z} = \iiint_E z\rho(x, y, z) \, dV.$$

If the density is constant, the center of mass is also called **centroid**.

Problem 12 (Stewart, Exercises 16.4.22). Let D be a region bounded by a simple closed path C in the xy-plane. Argue that the coordinate of the centroid (\bar{x}, \bar{y}) of D are

$$\bar{x} = \frac{1}{2A} \oint_C x^2 dy$$
 and $\bar{y} = \frac{1}{2A} \oint_C y^2 dx$

where A is the area of D. [Hint: Use Green's Theorem and $\rho = m/A$.]

Problem 13 (Cal Final, Summer 2018W). Let f be a differentiable function on \mathbb{R}^2 such that

$$\frac{\partial f}{\partial x}(x,y) = \frac{\partial f}{\partial y}(x,y)$$

for all x, y. Suppose also that f(2,3) = 6. Compute f(4,1).

2. Surface Integrals and Flux

Problem 14 (Stewart, Exercises 16.5.(13,15,18)). Determine whether or not the following vector fields are conservative. For each conservative, find a function f such that $F = \nabla f$.

(1) $F(x, y, z) = \langle y^2 z^3, 2xyz^3, 3xy^2 z^2 \rangle.$ (2) $F(x, y, z) = \langle z \cos y, 2xyz^3, xz \sin y, x \cos y \rangle.$ (3) $F(x, y, z) = \langle e^x \sin yz, ze^x \cos yz, ye^x \cos yz \rangle.$

Problem 15 (Stewart, Exercises 16.5.(19,20)). Is there a vector field G on \mathbb{R}^3 satisfying the following condition.

(1) $\operatorname{curl}(G) = \langle x \sin y, \cos y, z - xy \rangle$. Explain why? (2) $\operatorname{curl}(G) = \langle x, y, z \rangle$. Explain why?

The **Laplace operator** ∇^2 is defined by $\nabla^2 f = f_{xx} + f_{yy} + f_{zz}$, where f is a function having continuous second partial derivatives. The Laplace operator can be applied also to a vector field $F = \langle P, Q, R \rangle$ as follows: $\nabla^2 F = \langle \nabla^2 P, \nabla^2 Q, \nabla^2 R \rangle$.

Problem 16 (Stewart, Exercises 16.5.(27,28,29)). For vector fields F and G on \mathbb{R}^3 , argue the following identities:

- (1) $\operatorname{div}(F \times G) = G \cdot \operatorname{curl} F F \cdot \operatorname{curl} G$,
- (2) div $(\nabla F \times \nabla G) = 0$,
- (3) $\operatorname{curl}(\operatorname{curl} F) = \operatorname{grad}(\operatorname{div} F) \nabla^2 F.$

Problem 17 (Stewart, Exercises 16.5.(30,31)). For the vector field $r = \langle x, y, z \rangle$, argue the following identities:

- (1) $\nabla \cdot (|r|r) = 4|r|,$
- (2) $\nabla^2(|r|^3) = 12r$,
- (3) $\nabla(1/|r|) = -r/|r|^3$,
- (4) $\nabla(\ln |r|) = r/|r^2|.$

Problem 18 (Stewart, Exercises 16.5.(33,34)). Let C be a positively-oriented, piecewisesmooth, simple closed curve in \mathbb{R}^2 given by $r(t) = \langle x(t), y(t) \rangle$ with $a \leq t \leq b$, and let $n(t) = 1/|r'(t)| \langle y'(t), -x'(t) \rangle$. (1) Use Green's Theorem to argue that

$$\oint_C F \cdot n \, ds = \iint_{\operatorname{int}(C)} \operatorname{div} F(x, y) \, dA,$$

for any smooth vector field F on \mathbb{R}^2 .

(2) Use the previous part to argue Green's First Identity:

$$\iint_{\operatorname{int}(C)} f \nabla^2 g \, dA = \oint_C f(\nabla g) \cdot n \, ds - \iint_{\operatorname{int}(C)} \nabla f \cdot \nabla g \, dA,$$

for any functions f and g whose appropriate partial derivatives exist and are continuous.

(3) Use the previous part to argue Green's Second Identity:

$$\iint_{\operatorname{int}(C)} (f\nabla^2 g - g\nabla^2 f) \, dA = \oint_C (f\nabla g - g\nabla f) \cdot n \, ds,$$

for any functions f and g whose appropriate partial derivatives exist and are continuous.

Problem 19 (Stewart, Exercise 16.6.34). Find the equation of the tangent plane to the surface parameterized by $r(u, v) = (u^2 + 1, v^3 + 1, u + v)$ at (5, 2, 3).

Problem 20 (Cal Final, Summer 2018W). Find the tangent plane to the parametrized surface $r(u, v) = \langle u^2 - 1, uv, v^3 \rangle$ at the point (3, 4, 8).

Problem 21 (Stewart, Exercise 16.6.61). Find the area of the part of the sphere $x^2 + y^2 + z^2 = 4z$ that lies inside the paraboloid $z = x^2 + y^2$.

Problem 22. Deduce the formula for the flux of a vector field $F = \langle P, Q, R \rangle$ across a surface S that is the graph of a smooth function $g: U \subset \mathbb{R}^2 \to \mathbb{R}$.

- **Problem 23** (Stewart, Exercises 16.7.(15,26)). Evaluate the following surface integrals.
 - (1) $\iint_{S} y \, dS$, where S is the surface $y = x^2 + 4z$ for $(x, z) \in [0, 1] \times [0, 1]$.
 - (2) $\iint_S F \cdot dS$, where $F(x, y, z) = \langle y, -x, 2z \rangle$ and S is the hemisphere $x^2 + y^2 + z^2 = 4$ and $z \ge 0$ oriented downward.

Problem 24 (Stewart, Exercise 16.7.43). A fluid has density 870 kg/m³ and flows with velocity $v(x, y, z) = \langle z, y^2, x^2 \rangle$, where x, y, and z are measured in meters and the components of v in meters per second. Find the rate of flow outward through the cylinder $x^2 + y^2 = 4$ for $0 \le z \le 1$. [Hint: the rate of flow outward is the flux $\iint_{S}(\rho v) \cdot n \, dS$.]

Problem 25 (Stewart, Exercise 16.7.49). Let F be an inverse square field, that is, $F(r) = cr/|r|^3$ for some constant c, where $r = \langle x, y, z \rangle$. Show that the flux of F across a sphere S with center the origin is independent of the radius of S.

Problem 26 (Cal Final, Fall 09). Evaluate the surface integral

$$\iint_{S} z \, dS,$$

where S is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$ and above the xy-plane. [Answer: 2π]

Problem 27 (Cal Final, Fall 09). Evaluate the flux

$$\iint_T F \cdot dS$$

of the vector field $F(x, y, z) = \langle -x, xy, zx \rangle$ across the triangle T with vertices (1, 0, 0), (0, 2, 0), and (0, 0, 2) with downward orientation. [Answer: 1/3]

Problem 28 (Cal Final, Summer 2018W). Let S be a surface which is contained in the plane z = x + y, oriented upward. Suppose that S has area 2018. Consider the constant vector field $F = \langle 3, 4, 5 \rangle$. Calculate the flux of F across the surface S. [Answer: $-4036/\sqrt{3}$]

3. Stokes' Theorem and Divergence Theorem

Problem 29 (Stewart, Example 16.8.1). Find the line integral of the vector field $F = \langle -y^2, x, z^2 \rangle$ over the curve C of intersection of the plane x + z = 2 and the cylinder $x^2 + y^2 = 1$ knowing that C is oriented counterclockwise when viewed from above. [Answer: π]

Problem 30 (Stewart, Example 16.8.1). Find the flux of the vector field $F = \langle xz, yz, xy \rangle$ across the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$ and above the xy-plane.

Problem 31 (Exercise 16.8.10). Evaluate $\int_C F \cdot dr$ below if C is oriented counterclockwise as viewed from above.

- (1) $F(x, y, z) = \langle xy, yz, zx \rangle$ and C is the boundary of the part of the paraboloid $z = 1 x^2 y^2$ in the first octant.
- (2) $F(x, y, z) = \langle 2y, xz, x + y \rangle$ and C is the intersection of the plane z = y + 2 and the cylinder $x^2 + y^2 = 1$.

Problem 32 (Stewart, Exercise 16.8.16). Let C be a simple closed smooth curve that lies in the plane x + y + z = 1. Show that the line integral

$$\int_C z\,dx - 2x\,dy + 3y\,dz$$

depends only on the area of the region enclosed by C and not on the shape of C or its location in the plane.

Problem 33 (Stewart, Exercise 16.8.17). A particle moves along line segments from the origin to the points (1,0,0), (1,2,1), (0,2,1), and back to the origin under the influence of the force field

$$F(x, y, z) = \langle z^2, 2xy, 4y^2 \rangle.$$

Find the work done.

Problem 34 (Stewart, Exercise 16.8.18). Evaluate

$$\int_C (y+\sin x)dx + (z^2+\cos y)dy + x^3dz,$$

where C is the curve with parametrization $r(t) = (\sin t, \cos t, \sin 2t)$ for $0 \le t \le 2\pi$.

Problem 35 (Stewart, Example 16.9.1). Fin the flux of the vector field $F(x, y, z) = \langle z, y, x \rangle$ over the unit sphere $x^2 + y^2 + z^2 = 1$. [Answer: $4\pi/3$]

Problem 36 (Stewart, Example 16.9.2). Find the flux of the vector field

$$F(x, y, z) = \langle xy, (y^2 + e^{xz^2}, \sin(xy)) \rangle$$

over the surface of the region bounded by $z = 1 - x^2$ and the planes z = 0, y = 0, and y + z = 2. [Answer: 184/35]

Problem 37 (Stewart, Exercise 16.9.24). Use the Divergence Theorem to evaluate

$$\iint_{S} (2x + 2y + z^2) \, dS$$

where S is the sphere $x^2 + y^2 + z^2 = 1$.

Problem 38 (Stewart, Exercise 16.9.18). Find the flux of the vector field

$$F(x, y, z) = \langle z \tan^{-1}(y^2), z^3 \ln(x^2 + 1), z \rangle$$

across the part of the paraboloid $z^2 + y^2 + z = 2$ that lies above the plane z = 1 and is oriented upward.

Problem 39 (Stewart, Exercises 16.9.(26,29)). Argue the following identities assuming that S and E satisfy the conditions of the Divergence Theorem and the scalar functions and the vector fields have continuous second-order partial derivatives.

- (1) $V(E) = \frac{1}{3} \iint_{S} F \cdot dS$, where $F(x, y, z) = \langle x, y, z \rangle$.
- (2) $\iint_{S} (f \nabla g) \cdot n \, dS = \iiint_{F} (f \nabla^{2} g + \nabla f \cdot \nabla g) \, dV.$

Problem 40 (Cal Final, Fall 09). Evaluate the flux

$$\iint_{S} F \cdot dS,$$

where

$$F(x, y, z) = \langle z^2 x + e^{z^2 - y^2}, y^3 / 3 + x^2 y + \sin(z + x^2), x^2 \rangle$$

and S is the top half of the sphere $x^2 + y^2 + z^2 = 1$ oriented upward. [Answer: 13/20]

Problem 41 (Cal Final, Fall 09). A fisherman's net has a rim, which is a circle of radius 5. He fixes it in the sea in such a way that the rim is in the xz-plane with center at the origin. The velocity of water is given by the vector field

$$F(x, y, z) = \langle x^4 + 2y^2, 3 - y^2, 2yz - 4x^3z \rangle.$$

Find the flux of the water across the net. [Answer: 75π]

Problem 42 (Cal Final, Fall 09). Let S_r denote the sphere of radius r with the center at the origin and outward orientation. Suppose that E is a vector field well-defined on all of \mathbb{R}^3 and such that

$$\iint_{S_r} E \, dS = ar + b,$$

for some fixed constant a and b.

(1) Compute in terms of a and b the following integral

$$\iiint_D \operatorname{div} E \, dV,$$

where $D = \{(x, y, z) \in \mathbb{R}^3 \mid 25 \le x^2 + y^2 + z^2 \le 49\}$. [Answer: 2a]

(2) Suppose that in the above situation $E = \operatorname{curl} F$ for some vector field F. What conditions, if any, does this place on a and b?

[Answer: a = b = 0]

Problem 43 (Stewart, Exercise 16.9.31). Suppose that S and E satisfy the conditions of the Divergence Theorem and f is a scalar function having second-partial derivatives. Prove that

$$\iint_{S} fn \, dS = \iiint_{E} \nabla f \, dV.$$

These surface and triple integrals of vector functions are vectors defined by integrating each component function. [Hint: Start by applying the Divergence Theorem to F = fc, where c is an arbitrary constant vector.]

Problem 44 (Cal Final, Summer 2018W). Let C be the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane z = 2x + 3y, oriented counterclockwise when viewed from above. Let

$$F = \langle x^{2018} + y, y^{2018} + z, z^{2018} + x \rangle.$$

Calculate $\int_C F \cdot dr$.

Problem 45 (Cal Final, Summer 2018W). Calculate the flux $\iint_S F \cdot dS$, where S is the hemisphere $x^2 + y^2 + z^2 = 1$ with $z \ge 0$ oriented upwards, and $F = \langle x + \sin y, y + \cos z, z + 1 \rangle$.

References

[1] J. Stewart: Calculus 8th Edition, Cengage Learning, Boston 2016.