Stokes' Theorem and Divergence Theorem

Problem 1 (Stewart, Example 16.8.1). *Find the line integral of the vector field* $F = \langle -y^2, x, z^2 \rangle$ *over the curve* C *of intersection of the plane* $x + z = 2$ *and the cylinder* $x^2 + y^2 = 1$ *knowing that* C *is oriented counterclockwise when viewed from above. [Answer:* π*]*

Problem 2 (Stewart, Example 16.8.1). *Find the flux of the vector field* $F = \langle xz, yz, xy \rangle$ *across the part of the sphere* $x^2 + y^2 + z^2 = 4$ *that lies inside the cylinder* $x^2 + y^2 = 1$ *and above the* xy*-plane.*

Problem 3 (Exercise 16.8.10). *Evaluate* $\int_C F \cdot dr$ *below if* C *is oriented counterclockwise as viewed from above.*

(1) $F(x, y, z) = \langle xy, yz, zx \rangle$ *and* C *is the boundary of the part of the paraboloid* $z = 1 - x^2 - y^2$ *in the first octant.*

(2) $F(x, y, z) = \langle 2y, xz, x + y \rangle$ *and* C *is the intersection of the plane* $z = y + 2$ *and the cylinder* $x^2 + y^2 = 1$*.*

Problem 4 (Stewart, Exercise 16.8.16). *Let* C *be a simple closed smooth curve that lies in the plane* $x + y + z = 1$*. Show that the line integral*

$$\int_C z\,dx - 2x\,dy + 3y\,dz$$

depends only on the area of the region enclosed by C *and not on the shape of* C *or its location in the plane.*

Problem 5 (Stewart, Exercise 16.8.17). *A particle moves along line segments from the origin to the points* $(1, 0, 0), (1, 2, 1), (0, 2, 1)$*, and back to the origin under the influence of the force field*

$$F(x, y, z) = \langle z^2, 2xy, 4y^2 \rangle.$$

Find the work done.

Problem 6 (Stewart, Exercise 16.8.18). *Evaluate*

$$\int_C (y + \sin x)dx + (z^2 + \cos y)dy + x^3 dz,$$

where C *is the curve with parametrization* $r(t) = (\sin t, \cos t, \sin 2t)$ *for* $0 \le t \le 2\pi$*.*

Problem 7 (Stewart, Example 16.9.1). *Fin the flux of the vector field* $F(x, y, z) = \langle z, y, x \rangle$ *over the unit sphere* $x^2 + y^2 + z^2 = 1$*. [Answer:* $4\pi/3$*]*

Problem 8 (Stewart, Example 16.9.2). *Find the flux of the vector field*

$$F(x,y,z) = \langle xy, (y^2 + e^{xz^2}, \sin(xy)\rangle$$

over the surface of the region bounded by $z = 1 - x^2$ *and the planes* $z = 0$, $y = 0$, *and* $y + z = 2$. *[Answer:* $184/35$*]*

Problem 9 (Stewart, Exercise 16.9.24). *Use the Divergence Theorem to evaluate*

$$\iint_S (2x + 2y + z^2)\, dS,$$

where S *is the sphere* $x^2 + y^2 + z^2 = 1$.

Problem 10 (Stewart, Exercise 16.9.18). *Find the flux of the vector field*

$$F(x,y,z) = \langle z\tan^{-1}(y^2), z^3\ln(x^2+1), z\rangle$$

across the part of the paraboloid $z^2 + y^2 + z = 2$ *that lies above the plane* $z = 1$ *and is oriented upward.*

Problem 11 (Stewart, Exercises 16.9.(26,29)). *Argue the following identities assuming that* S *and* E *satisfy the conditions of the Divergence Theorem and the scalar functions and the vector fields have continuous second-order partial derivatives.*

(1) $V(E) = \frac{1}{3}\iint_S F\cdot dS$, *where* $F(x,y,z) = \langle x,y,z\rangle$.

(2) $\iint_S (f\nabla g)\cdot n\, dS = \iiint_E (f\nabla^2 g + \nabla f\cdot\nabla g)\, dV$.

Problem 12 (UC Berkeley Final). *Evaluate the flux*

$$\iint_S F\cdot dS,$$

where

$$F(x,y,z) = \langle z^2x + e^{z^2-y^2}, y^3/3 + x^2y + \sin(z+x^2), x^2\rangle$$

and S *is the top half of the sphere* $x^2 + y^2 + z^2 = 1$ *oriented upward. [Answer:* $13/20$*]*

Problem 13 (UC Berkeley Final). *A fisherman's net has a rim, which is a circle of radius* 5. *He fixes it in the sea in such a way that the rim is in the* xz*-plane with center at the origin. The velocity of water is given by the vector field*

$$F(x,y,z) = \langle x^4 + 2y^2, 3 - y^2, 2yz - 4x^3z\rangle.$$

Find the flux of the water across the net. [Answer: 75π*]*

Problem 14 (UC Berkeley Final). *Let* S_r *denote the sphere of radius* r *with the center at the origin and outward orientation. Suppose that* E *is a vector field well-defined on all of* $\mathbb{R}^3$ *and such that*

$$\iint_{S_r} E\, dS = ar + b,$$

for some fixed constant a *and* b.

(1) Compute in terms of a and b the following integral

$$\iiint_D \operatorname{div} E \, dV,$$

where $D = \{(x, y, z) \in \mathbb{R}^3 \mid 25 \leq x^2 + y^2 + z^2 \leq 49\}$. [Answer: $2a$]

(2) Suppose that in the above situation $E = \operatorname{curl} F$ for some vector field F. What conditions, if any, does this place on a and b?

[Answer: $a = b = 0$]

Problem 15 (Stewart, Exercise 16.9.31)**.** *Suppose that S and E satisfy the conditions of the Divergence Theorem and f is a scalar function having second-partial derivatives. Prove that*

$$\iint_S f n \, dS = \iiint_E \nabla f \, dV.$$

These surface and triple integrals of vector functions are vectors defined by integrating each component function. [Hint: Start by applying the Divergence Theorem to $F = fc$, where c is an arbitrary constant vector.]

Problem 16 (UC Berkeley Final)**.** *Let C be the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $z = 2x + 3y$, oriented counterclockwise when viewed from above. Let*

$$F = \langle x^{2018} + y, y^{2018} + z, z^{2018} + x \rangle.$$

Calculate $\int_C F \cdot dr$.

Problem 17 (UC Berkeley Final)**.** *Calculate the flux $\iint_S F \cdot dS$, where S is the hemisphere $x^2 + y^2 + z^2 = 1$ with $z \geq 0$ oriented upwards, and $F = \langle x + \sin y, y + \cos z, z + 1 \rangle$.*

References

[1] J. Stewart: *Calculus* 8th Edition, Cengage Learning, Boston 2016.