Surface Integrals and Flux

Problem 1 (Stewart, Exercises 16.5.(13,15,18)). *Determine whether or not the following vector fields are conservative. For each conservative, find a function f such that $F = \nabla f$.*

(1) $F(x, y, z) = \langle y^2z^3, 2xyz^3, 3xy^2z^2\rangle$.

(2) $F(x, y, z) = \langle z\cos y, 2xyz^3, xz\sin y, x\cos y\rangle$.

(3) $F(x, y, z) = \langle e^x \sin yz, ze^x \cos yz, ye^x \cos yz\rangle$.

Problem 2 (Stewart, Exercises 16.5.(19,20)). *Is there a vector field G on $\mathbb{R}^3$ satisfying the following condition.*

(1) $\operatorname{curl}(G) = \langle x\sin y, \cos y, z - xy\rangle$. *Explain why?*

(2) $\operatorname{curl}(G) = \langle x, y, z\rangle$. *Explain why?*

The **Laplace operator** ∇^2 is defined by $\nabla^2 f = f_{xx} + f_{yy} + f_{zz}$, where f is a function having continuous second partial derivatives. The Laplace operator can be applied also to a vector field $F = \langle P, Q, R\rangle$ as follows: $\nabla^2 F = \langle\nabla^2 P, \nabla^2 Q, \nabla^2 R\rangle$.

Problem 3 (Stewart, Exercises 16.5.(27,28,29)). *For vector fields F and G on $\mathbb{R}^3$, argue the following identities:*

(1) $\operatorname{div}(F \times G) = G \cdot \operatorname{curl} F - F \cdot \operatorname{curl} G$,

(2) $\operatorname{div}(\nabla F \times \nabla G) = 0$,

(3) $\operatorname{curl}(\operatorname{curl} F) = \operatorname{grad}(\operatorname{div} F) - \nabla^2 F$.

Problem 4 (Stewart, Exercises 16.5.(30,31)). *For the vector field $r = \langle x, y, z\rangle$, argue the following identities:*

(1) $\nabla \cdot (|r|r) = 4|r|$,

(2) $\nabla^2(|r|^3) = 12r$,

(3) $\nabla(1/|r|) = -r/|r|^3$,

(4) $\nabla(\ln |r|) = r/|r^2|$.

Problem 5 (Stewart, Exercises 16.5.(33,34)). *Let C be a positively-oriented, piecewise-smooth, simple closed curve in $\mathbb{R}^2$ given by $r(t) = \langle x(t), y(t)\rangle$ with $a \le t \le b$, and let $n(t) = 1/|r'(t)|\langle y'(t), -x'(t)\rangle$.*

(1) Use Green's Theorem to argue that

$$\oint_C F \cdot n \, ds = \iint_{\operatorname{int}(C)} \operatorname{div} F(x, y)\, dA,$$

for any smooth vector field F on $\mathbb{R}^2$.

(2) *Use the previous part to argue Green's First Identity:*

$$\iint_{\mathrm{int}(C)} f\nabla^2 g\, dA = \oint_C f(\nabla g)\cdot n\, ds - \iint_{\mathrm{int}(C)} \nabla f\cdot\nabla g\, dA,$$

for any functions f and g whose appropriate partial derivatives exist and are continuous.

(3) *Use the previous part to argue Green's Second Identity:*

$$\iint_{\mathrm{int}(C)} (f\nabla^2 g - g\nabla^2 f)\, dA = \oint_C (f\nabla g - g\nabla f)\cdot n\, ds,$$

for any functions f and g whose appropriate partial derivatives exist and are continuous.

Problem 6 (Stewart, Exercise 16.6.34). *Find the equation of the tangent plane to the surface parameterized by $r(u,v) = (u^2+1, v^3+1, u+v)$ at $(5,2,3)$.*

Problem 7 (Cal Final, Summer 2018W). *Find the tangent plane to the parametrized surface $r(u,v) = \langle u^2-1, uv, v^3\rangle$ at the point $(3,4,8)$.*

Problem 8 (Stewart, Exercise 16.6.61). *Find the area of the part of the sphere $x^2+y^2+z^2=4z$ that lies inside the paraboloid $z=x^2+y^2$.*

Problem 9. *Deduce the formula for the flux of a vector field $F=\langle P,Q,R\rangle$ across a surface S that is the graph of a smooth function $g\colon U\subset\mathbb{R}^2\to\mathbb{R}$.*

Problem 10 (Stewart, Exercises 16.7.(15,26)). *Evaluate the following surface integrals.*

(1) $\iint_S y\, dS$*, where S is the surface $y=x^2+4z$ for $(x,z)\in[0,1]\times[0,1]$.*

(2) $\iint_S F\cdot dS$*, where $F(x,y,z)=\langle y,-x,2z\rangle$ and S is the hemisphere $x^2+y^2+z^2=4$ and $z\geq 0$ oriented downward.*

Problem 11 (Stewart, Exercise 16.7.43). *A fluid has density 870 kg/m^3 and flows with velocity $v(x,y,z)=\langle z,y^2,x^2\rangle$, where $x,y,$ and z are measured in meters and the components of v in meters per second. Find the rate of flow outward through the cylinder $x^2+y^2=4$ for $0\leq z\leq 1$. [Hint: the rate of flow outward is the flux $\iint_S(\rho v)\cdot n\, dS$.]*

Problem 12 (Stewart, Exercise 16.7.49). *Let F be an inverse square field, that is, $F(r)=cr/|r|^3$ for some constant c, where $r=\langle x,y,z\rangle$. Show that the flux of F across a sphere S with center the origin is independent of the radius of S.*

Problem 13 (Cal Final, Fall 09). *Evaluate the surface integral*

$$\iint_S z\, dS,$$

where S is the part of the sphere $x^2+y^2+z^2=4$ that lies inside the cylinder $x^2+y^2=1$ and above the xy-plane. [Answer: 2π]

Problem 14 (UC Berkeley Final)**.** *Evaluate the flux*

$$\iint_T F \cdot dS$$

of the vector field $F(x, y, z) = \langle -x, xy, zx \rangle$ *across the triangle* T *with vertices* $(1, 0, 0)$, $(0, 2, 0)$, *and* $(0, 0, 2)$ *with downward orientation. [Answer:* $1/3$*]*

Problem 15 (UC Berkeley Final)**.** *Let* S *be a surface which is contained in the plane* $z = x+y$, *oriented upward. Suppose that* S *has area* 2018. *Consider the constant vector field* $F = \langle 3, 4, 5 \rangle$. *Calculate the flux of* F *across the surface* S. *[Answer:* $-4036/\sqrt{3}$*]*

References

[1] J. Stewart: *Calculus* 8th Edition, Cengage Learning, Boston 2016.