Line Integrals and Green's Theorem

Problem 1 (Stewart, Exercise 16.1.(25,26)). *Find and sketch the gradient vector field of the following functions:*

$$(1)\ f(x,y) = \frac{1}{2}(x-y)^2 \qquad\qquad (2)\ f(x,y) = \frac{1}{2}(x^2-y^2).$$

Problem 2 (Stewart, Exercise 16.2.(5,11,14)). *Evaluate the following line integrals:*

(1) $\int_C (x^2y + \sin x)\,dy$, *where* C *is the arc of the parabola* $y = x^2$ *from* $(0,0)$ *to* (π, π^2);

(2) $\int_C xe^{yz}\,ds$, *where* C *is the line segment from* $(0,0,0)$ *to* $(1,2,3)$;

(3) $\int_C y\,dx + z\,dyx\,dz$, *where* $C = (\sqrt{t}, t, t^2)$ *for* $1 \le t \le 4$.

Problem 3 (Stewart, Exercise 16.2.41). *Find the work done by the force field*

$$F(x,y,z) = \langle x - y^2, y - z^2, z - x^2\rangle$$

on a particle that moves along the line segment from $(0,0,1)$ *to* $(2,1,0)$.

Problem 4 (Stewart, Exercise 16.2.(5,11,14)). *Let* C *be a smooth curve given by a vector function* $r(t)$ *for* $a \le t \le b$, *and let* v *be a constant vector*

(1) *Show that* $\int_C v \cdot dr = v \cdot (r(b) - r(a))$.

(2) *Show that* $\int_C r \cdot dr = \frac{1}{2}\big(|r(b)|^2 - |r(a)|^2\big)$.

Problem 5 (Stewart, Exercise 16.3.(14,18)). *Find a function* f *such that* $F = \nabla f$ *and use it to evaluate* $\int_C F \cdot dr$ *along the given curve.*

(1) $F(x,y) = \langle (1+xy)e^{xy}, x^2e^{xy}\rangle$, *where* $C : r(t) = \langle \cos t, 2\sin t\rangle$ *for* $0 \le t \le \pi/2$.

(2) $F(x,y,z) = \langle \sin y, x\cos y + \cos z, -y\sin z\rangle$, *where* $C : r(t) = \langle \sin t, t, 2t\rangle$ *for* $0 \le t \le \pi/2$.

Problem 6 (Stewart, Exercise 16.3.(19,20)). *Show that the following line integrals are independent of paths evaluate them.*

(1) $\int_C 2xe^{-y}\,dx + (2y - x^2e^{-y})\,dy$, *where* C *is any path from* $(1,0)$ *to* $(2,1)$.

(2) $\int_C \sin y\,dx + (x\cos y - \sin y)\,dy$, *where* C *is any path from* $(2,0)$ *to* $(1,\pi)$.

Problem 7 (Stewart, Exercise 16.3.(19,20)). *Determine whether or not the given subsets of* $\mathbb{R}^2$ *are open, connected, and/or simply connected:*

(1) $\{(x,y) \mid 0 < y < 3\}$;

(2) $\{(x,y) \mid 1 < |x| < y\}$;

(3) $\{(x,y) \mid 1 \le x^2 + y^2 \le 4, y \ge 0\}$;

(4) $\{(x,y) \mid (x,y) \ne (2,3)\}$.

Problem 8 (UC Berkeley Final). *Suppose a parametrized curve*

$$r(t) = \langle x(t), y(t), z(t)\rangle \quad \textit{for} \quad 0 \le t \le 1$$

satisfies the equation $xx'(t) + yy'(t) + zz'(t) = 0$ *for all* t. *If* $x(0) = y(0) = z(0) = 3$ *and* $x(1) = y(1) = 2$, *find* $|z(1)|$.

Problem 9 (Stewart, Exercises 16.4.(1,7)). *Evaluate the following integral via Green's Theorem:*

(1) $\oint_C y^2\,dx + x^2y\,dy$, *where* C *is the rectangle with vertices* $(0,0)$, $(5,0)$, $(5,4)$, *and* $(0,4)$;

(2) $\int_C (y+e^{\sqrt{x}})\,dx + (2x+\cos y^2)\,dy$, *where* C *is the boundary of the region enclosed by the parabolas* $y = x^2$ *and* $x = y^2$.

Problem 10 (Stewart, Exercises 16.4.18). *A particle starts at the origin, move along the* x*-axis to* $(5,0)$, *then along the quarter-circle* $x^2 + y^2 = 25$ *where* $x \ge 0$ *and* $y \ge 0$ *to the point* $(0,5)$, *and then down to the* y*-axis back to the origin. Find the work done on this particle by the force field* $F = \langle \sin x, \sin y + xy^2 + \frac{1}{3}x^3\rangle$.

Problem 11 (Stewart, Exercises 16.4.19). *Let* D *be the region bounded by a positively-oriented, piecewise-smooth, simple closed curve* C.

(1) *Argue that* $\text{Area}(D) = \oint_C x\,dy = -\oint_C y\,dx = 1/2\oint_C x\,dy - y\,dx$.

(2) *Use the previous part to find the area under one arch of the cycloid* $(t - \sin t, 1 - \cos t)$.

If the **density** of a solid occupying the region E is given by $\rho(x,y,z)$, then its mass can be computed by

$$m = \iiint_E \rho(x,y,z)\,dV$$

and its **center of mass** is $(\bar{x}, \bar{y}, \bar{z})$, where

$$\bar{x} = \iiint_E x\rho(x,y,z)\,dV; \quad \bar{y} = \iiint_E y\rho(x,y,z)\,dV; \quad \bar{z} = \iiint_E z\rho(x,y,z)\,dV.$$

If the density is constant, the center of mass is also called **centroid**.

Problem 12 (Stewart, Exercises 16.4.22). *Let* D *be a region bounded by a simple closed path* C *in the* xy*-plane. Argue that the coordinate of the centroid* $(\bar{x}, \bar{y})$ *of* D *are*

$$\bar{x} = \frac{1}{2A}\oint_C x^2\,dy \quad \textit{and} \quad \bar{y} = \frac{1}{2A}\oint_C y^2\,dx,$$

where A *is the area of* D. *[Hint: Use Green's Theorem and* $\rho = m/A$*.]*

Problem 13 (UC Berkeley Final)**.** *Let f be a differentiable function on $\mathbb{R}^2$ such that*

$$\frac{\partial f}{\partial x}(x, y) = \frac{\partial f}{\partial y}(x, y)$$

for all x, y. Suppose also that $f(2, 3) = 6$. Compute $f(4, 1)$.

References

[1] J. Stewart: *Calculus* 8th Edition, Cengage Learning, Boston 2016.