MULTIVARIABLE CALCULUS

1. Line Integrals and Green's Theorem

Problem 1 (Stewart, Exercise 16.1.(25,26)). *Find and sketch the gradient vector field of the following functions:*

$$(1)\ f(x,y) = \frac{1}{2}(x-y)^2 \qquad (2)\ f(x,y) = \frac{1}{2}(x^2-y^2).$$

Problem 2 (Stewart, Exercise 16.2.(5,11,14)). *Evaluate the following line integrals:*

(1) $\int_C (x^2y + \sin x)\,dy$, *where* C *is the arc of the parabola* $y = x^2$ *from* $(0,0)$ *to* (π, π^2);

(2) $\int_C xe^{yz}\,ds$, *where* C *is the line segment from* $(0,0,0)$ *to* $(1,2,3)$;

(3) $\int_C y\,dx + z\,dyx\,dz$, *where* $C = (\sqrt{t}, t, t^2)$ *for* $1 \le t \le 4$.

Problem 3 (Stewart, Exercise 16.2.41). *Find the work done by the force field*

$$F(x,y,z) = \langle x - y^2, y - z^2, z - x^2 \rangle$$

on a particle that moves along the line segment from $(0,0,1)$ *to* $(2,1,0)$.

Problem 4 (Stewart, Exercise 16.2.(5,11,14)). *Let* C *be a smooth curve given by a vector function* $r(t)$ *for* $a \le t \le b$, *and let* v *be a constant vector*

(1) *Show that* $\int_C v \cdot dr = v \cdot (r(b) - r(a))$.

(2) *Show that* $\int_C r \cdot dr = \frac{1}{2}\big(|r(b)|^2 - |r(a)|^2\big)$.

Problem 5 (Stewart, Exercise 16.3.(14,18)). *Find a function* f *such that* $F = \nabla f$ *and use it to evaluate* $\int_C F \cdot dr$ *along the given curve.*

(1) $F(x,y) = \langle (1+xy)e^{xy}, x^2e^{xy} \rangle$, *where* $C : r(t) = \langle \cos t, 2\sin t \rangle$ *for* $0 \le t \le \pi/2$.

(2) $F(x,y,z) = \langle \sin y, x\cos y + \cos z, -y\sin z \rangle$, *where* $C : r(t) = \langle \sin t, t, 2t \rangle$ *for* $0 \le t \le \pi/2$.

Problem 6 (Stewart, Exercise 16.3.(19,20)). *Show that the following line integrals are independent of paths evaluate them.*

(1) $\int_C 2xe^{-y}\,dx + (2y - x^2e^{-y})\,dy$, *where* C *is any path from* $(1,0)$ *to* $(2,1)$.

(2) $\int_C \sin y\,dx + (x\cos y - \sin y)\,dy$, *where* C *is any path from* $(2,0)$ *to* $(1,\pi)$.

Problem 7 (Stewart, Exercise 16.3.(19,20)). *Determine whether or not the given subsets of* $\mathbb{R}^2$ *are open, connected, and/or simply connected:*

(1) $\{(x,y) \mid 0 < y < 3\}$;

(2) $\{(x, y) \mid 1 < |x| < y\}$*;*

(3) $\{(x, y) \mid 1 \leq x^2 + y^2 \leq 4, y \geq 0\}$*;*

(4) $\{(x, y) \mid (x, y) \neq (2, 3)\}$*.*

Problem 8 (Cal Final, Summer 2018W)**.** *Suppose a parametrized curve*

$$r(t) = \langle x(t), y(t), z(t)\rangle \quad \textit{for} \quad 0 \leq t \leq 1$$

satisfies the equation $xx'(t) + yy'(t) + zz'(t) = 0$ *for all* t*. If* $x(0) = y(0) = z(0) = 3$ *and* $x(1) = y(1) = 2$*, find* $|z(1)|$*.*

Problem 9 (Stewart, Exercises 16.4.(1,7))**.** *Evaluate the following integral via Green's Theorem:*

(1) $\oint_C y^2\,dx + x^2 y\,dy$*, where* C *is the rectangle with vertices* $(0, 0)$*,* $(5, 0)$*,* $(5, 4)$*, and* $(0, 4)$*;*

(2) $\int_C (y + e^{\sqrt{x}})\,dx + (2x + \cos y^2)\,dy$*, where* C *is the boundary of the region enclosed by the parabolas* $y = x^2$ *and* $x = y^2$*.*

Problem 10 (Stewart, Exercises 16.4.18)**.** *A particle starts at the origin, move along the* x*-axis to* $(5, 0)$*, then along the quarter-circle* $x^2 + y^2 = 25$ *where* $x \geq 0$ *and* $y \geq 0$ *to the point* $(0, 5)$*, and then down to the* y*-axis back to the origin. Find the work done on this particle by the force field* $F = \langle \sin x, \sin y + xy^2 + \frac{1}{3}x^3\rangle$*.*

Problem 11 (Stewart, Exercises 16.4.19)**.** *Let* D *be the region bounded by a positively-oriented, piecewise-smooth, simple closed curve* C*.*

(1) Argue that $\text{Area}(D) = \oint_C x\,dy = -\oint_C y\,dx = 1/2 \oint_C x\,dy - y\,dx$*.*

(2) Use the previous part to find the area under one arch of the cycloid $(t - \sin t, 1 - \cos t)$*.*

If the **density** of a solid occupying the region E is given by $\rho(x, y, z)$, then its mass can be computed by

$$m = \iiint_E \rho(x, y, z)\,dV$$

and its **center of mass** is $(\bar{x}, \bar{y}, \bar{z})$, where

$$\bar{x} = \iiint_E x\rho(x, y, z)\,dV; \quad \bar{y} = \iiint_E y\rho(x, y, z)\,dV; \quad \bar{z} = \iiint_E z\rho(x, y, z)\,dV.$$

If the density is constant, the center of mass is also called **centroid**.

Problem 12 (Stewart, Exercises 16.4.22)**.** *Let* D *be a region bounded by a simple closed path* C *in the* xy*-plane. Argue that the coordinate of the centroid* $(\bar{x}, \bar{y})$ *of* D *are*

$$\bar{x} = \frac{1}{2A}\oint_C x^2\,dy \quad \textit{and} \quad \bar{y} = \frac{1}{2A}\oint_C y^2\,dx,$$

where A *is the area of* D*. [Hint: Use Green's Theorem and* $\rho = m/A$*.]*

Problem 13 (Cal Final, Summer 2018W). *Let f be a differentiable function on $\mathbb{R}^2$ such that*

$$\frac{\partial f}{\partial x}(x,y) = \frac{\partial f}{\partial y}(x,y)$$

for all x, y. Suppose also that $f(2,3) = 6$. Compute $f(4,1)$.

2. Surface Integrals and Flux

Problem 14 (Stewart, Exercises 16.5.(13,15,18)). *Determine whether or not the following vector fields are conservative. For each conservative, find a function f such that $F = \nabla f$.*

(1) $F(x,y,z) = \langle y^2z^3, 2xyz^3, 3xy^2z^2\rangle$.

(2) $F(x,y,z) = \langle z\cos y, 2xyz^3, xz\sin y, x\cos y\rangle$.

(3) $F(x,y,z) = \langle e^x \sin yz, ze^x \cos yz, ye^x \cos yz\rangle$.

Problem 15 (Stewart, Exercises 16.5.(19,20)). *Is there a vector field G on $\mathbb{R}^3$ satisfying the following condition.*

(1) $\mathrm{curl}(G) = \langle x\sin y, \cos y, z - xy\rangle$. *Explain why?*

(2) $\mathrm{curl}(G) = \langle x, y, z\rangle$. *Explain why?*

The **Laplace operator** ∇^2 is defined by $\nabla^2 f = f_{xx} + f_{yy} + f_{zz}$, where f is a function having continuous second partial derivatives. The Laplace operator can be applied also to a vector field $F = \langle P, Q, R\rangle$ as follows: $\nabla^2 F = \langle \nabla^2 P, \nabla^2 Q, \nabla^2 R\rangle$.

Problem 16 (Stewart, Exercises 16.5.(27,28,29)). *For vector fields F and G on $\mathbb{R}^3$, argue the following identities:*

(1) $\mathrm{div}(F \times G) = G \cdot \mathrm{curl}\, F - F \cdot \mathrm{curl}\, G$,

(2) $\mathrm{div}(\nabla F \times \nabla G) = 0$,

(3) $\mathrm{curl}(\mathrm{curl}\, F) = \mathrm{grad}(\mathrm{div}\, F) - \nabla^2 F$.

Problem 17 (Stewart, Exercises 16.5.(30,31)). *For the vector field $r = \langle x, y, z\rangle$, argue the following identities:*

(1) $\nabla \cdot (|r|r) = 4|r|$,

(2) $\nabla^2(|r|^3) = 12r$,

(3) $\nabla(1/|r|) = -r/|r|^3$,

(4) $\nabla(\ln |r|) = r/|r^2|$.

Problem 18 (Stewart, Exercises 16.5.(33,34)). *Let C be a positively-oriented, piecewise-smooth, simple closed curve in $\mathbb{R}^2$ given by $r(t) = \langle x(t), y(t)\rangle$ with $a \le t \le b$, and let $n(t) = 1/|r'(t)|\langle y'(t), -x'(t)\rangle$.*

(1) *Use Green's Theorem to argue that*

$$\oint_C F \cdot n\, ds = \iint_{\mathrm{int}(C)} \operatorname{div} F(x,y)\, dA,$$

for any smooth vector field F on $\mathbb{R}^2$.

(2) *Use the previous part to argue Green's First Identity:*

$$\iint_{\mathrm{int}(C)} f\nabla^2 g\, dA = \oint_C f(\nabla g)\cdot n\, ds - \iint_{\mathrm{int}(C)} \nabla f \cdot \nabla g\, dA,$$

for any functions f and g whose appropriate partial derivatives exist and are continuous.

(3) *Use the previous part to argue Green's Second Identity:*

$$\iint_{\mathrm{int}(C)} (f\nabla^2 g - g\nabla^2 f)\, dA = \oint_C (f\nabla g - g\nabla f)\cdot n\, ds,$$

for any functions f and g whose appropriate partial derivatives exist and are continuous.

Problem 19 (Stewart, Exercise 16.6.34)**.** *Find the equation of the tangent plane to the surface parameterized by $r(u,v) = (u^2+1, v^3+1, u+v)$ at $(5,2,3)$.*

Problem 20 (Cal Final, Summer 2018W)**.** *Find the tangent plane to the parametrized surface $r(u,v) = \langle u^2-1, uv, v^3\rangle$ at the point $(3,4,8)$.*

Problem 21 (Stewart, Exercise 16.6.61)**.** *Find the area of the part of the sphere $x^2 + y^2 + z^2 = 4z$ that lies inside the paraboloid $z = x^2 + y^2$.*

Problem 22. *Deduce the formula for the flux of a vector field $F = \langle P, Q, R\rangle$ across a surface S that is the graph of a smooth function $g\colon U \subset \mathbb{R}^2 \to \mathbb{R}$.*

Problem 23 (Stewart, Exercises 16.7.(15,26))**.** *Evaluate the following surface integrals.*

(1) $\iint_S y\, dS$, *where S is the surface $y = x^2 + 4z$ for $(x,z) \in [0,1]\times[0,1]$.*

(2) $\iint_S F\cdot dS$, *where $F(x,y,z) = \langle y, -x, 2z\rangle$ and S is the hemisphere $x^2+y^2+z^2 = 4$ and $z \geq 0$ oriented downward.*

Problem 24 (Stewart, Exercise 16.7.43)**.** *A fluid has density 870 kg/m^3 and flows with velocity $v(x,y,z) = \langle z, y^2, x^2\rangle$, where x, y, and z are measured in meters and the components of v in meters per second. Find the rate of flow outward through the cylinder $x^2 + y^2 = 4$ for $0 \leq z \leq 1$. [Hint: the rate of flow outward is the flux $\iint_S (\rho v)\cdot n\, dS$.]*

Problem 25 (Stewart, Exercise 16.7.49)**.** *Let F be an inverse square field, that is, $F(r) = cr/|r|^3$ for some constant c, where $r = \langle x,y,z\rangle$. Show that the flux of F across a sphere S with center the origin is independent of the radius of S.*

Problem 26 (Cal Final, Fall 09)**.** *Evaluate the surface integral*

$$\iint_S z\,dS,$$

where S is the part of the sphere $x^2+y^2+z^2=4$ that lies inside the cylinder $x^2+y^2=1$ and above the xy-plane. [Answer: 2π]

Problem 27 (Cal Final, Fall 09)**.** *Evaluate the flux*

$$\iint_T F\cdot dS$$

of the vector field $F(x,y,z)=\langle -x, xy, zx\rangle$ across the triangle T with vertices $(1,0,0)$, $(0,2,0)$, and $(0,0,2)$ with downward orientation. [Answer: $1/3$]

Problem 28 (Cal Final, Summer 2018W)**.** *Let S be a surface which is contained in the plane $z=x+y$, oriented upward. Suppose that S has area 2018. Consider the constant vector field $F=\langle 3,4,5\rangle$. Calculate the flux of F across the surface S. [Answer: $-4036/\sqrt{3}$]*

3. Stokes' Theorem and Divergence Theorem

Problem 29 (Stewart, Example 16.8.1)**.** *Find the line integral of the vector field $F=\langle -y^2, x, z^2\rangle$ over the curve C of intersection of the plane $x+z=2$ and the cylinder $x^2+y^2=1$ knowing that C is oriented counterclockwise when viewed from above. [Answer: π]*

Problem 30 (Stewart, Example 16.8.1)**.** *Find the flux of the vector field $F=\langle xz, yz, xy\rangle$ across the part of the sphere $x^2+y^2+z^2=4$ that lies inside the cylinder $x^2+y^2=1$ and above the xy-plane.*

Problem 31 (Exercise 16.8.10)**.** *Evaluate $\int_C F\cdot dr$ below if C is oriented counterclockwise as viewed from above.*

(1) $F(x,y,z)=\langle xy, yz, zx\rangle$ and C is the boundary of the part of the paraboloid $z=1-x^2-y^2$ in the first octant.

(2) $F(x,y,z)=\langle 2y, xz, x+y\rangle$ and C is the intersection of the plane $z=y+2$ and the cylinder $x^2+y^2=1$.

Problem 32 (Stewart, Exercise 16.8.16)**.** *Let C be a simple closed smooth curve that lies in the plane $x+y+z=1$. Show that the line integral*

$$\int_C z\,dx - 2x\,dy + 3y\,dz$$

depends only on the area of the region enclosed by C and not on the shape of C or its location in the plane.

Problem 33 (Stewart, Exercise 16.8.17). *A particle moves along line segments from the origin to the points* $(1, 0, 0), (1, 2, 1), (0, 2, 1)$, *and back to the origin under the influence of the force field*

$$F(x, y, z) = \langle z^2, 2xy, 4y^2 \rangle.$$

Find the work done.

Problem 34 (Stewart, Exercise 16.8.18). *Evaluate*

$$\int_C (y + \sin x)dx + (z^2 + \cos y)dy + x^3 dz,$$

where C *is the curve with parametrization* $r(t) = (\sin t, \cos t, \sin 2t)$ *for* $0 \leq t \leq 2\pi$.

Problem 35 (Stewart, Example 16.9.1). *Fin the flux of the vector field* $F(x, y, z) = \langle z, y, x \rangle$ *over the unit sphere* $x^2 + y^2 + z^2 = 1$. *[Answer:* $4\pi/3$*]*

Problem 36 (Stewart, Example 16.9.2). *Find the flux of the vector field*

$$F(x, y, z) = \langle xy, (y^2 + e^{xz^2}, \sin(xy) \rangle$$

over the surface of the region bounded by $z = 1 - x^2$ *and the planes* $z = 0$, $y = 0$, *and* $y + z = 2$. *[Answer:* $184/35$*]*

Problem 37 (Stewart, Exercise 16.9.24). *Use the Divergence Theorem to evaluate*

$$\iint_S (2x + 2y + z^2)\, dS,$$

where S *is the sphere* $x^2 + y^2 + z^2 = 1$.

Problem 38 (Stewart, Exercise 16.9.18). *Find the flux of the vector field*

$$F(x, y, z) = \langle z \tan^{-1}(y^2), z^3 \ln(x^2 + 1), z \rangle$$

across the part of the paraboloid $z^2 + y^2 + z = 2$ *that lies above the plane* $z = 1$ *and is oriented upward.*

Problem 39 (Stewart, Exercises 16.9.(26,29)). *Argue the following identities assuming that* S *and* E *satisfy the conditions of the Divergence Theorem and the scalar functions and the vector fields have continuous second-order partial derivatives.*

(1) $V(E) = \frac{1}{3} \iint_S F \cdot dS$, *where* $F(x, y, z) = \langle x, y, z \rangle$.

(2) $\iint_S (f \nabla g) \cdot n\, dS = \iiint_E (f \nabla^2 g + \nabla f \cdot \nabla g)\, dV$.

Problem 40 (Cal Final, Fall 09). *Evaluate the flux*

$$\iint_S F \cdot dS,$$

where

$$F(x, y, z) = \langle z^2 x + e^{z^2 - y^2}, y^3/3 + x^2 y + \sin(z + x^2), x^2 \rangle$$

and S *is the top half of the sphere* $x^2 + y^2 + z^2 = 1$ *oriented upward. [Answer:* $13/20$*]*

Problem 41 (Cal Final, Fall 09). *A fisherman's net has a rim, which is a circle of radius* 5. *He fixes it in the sea in such a way that the rim is in the* xz*-plane with center at the origin. The velocity of water is given by the vector field*

$$F(x, y, z) = \langle x^4 + 2y^2, 3 - y^2, 2yz - 4x^3 z\rangle.$$

Find the flux of the water across the net. [Answer: 75π*]*

Problem 42 (Cal Final, Fall 09). *Let* S_r *denote the sphere of radius* r *with the center at the origin and outward orientation. Suppose that* E *is a vector field well-defined on all of* $\mathbb{R}^3$ *and such that*

$$\iint_{S_r} E\, dS = ar + b,$$

for some fixed constant a *and* b.

(1) *Compute in terms of* a *and* b *the following integral*

$$\iiint_D \operatorname{div} E\, dV,$$

where $D = \{(x, y, z) \in \mathbb{R}^3 \mid 25 \leq x^2 + y^2 + z^2 \leq 49\}$. *[Answer:* $2a$*]*

(2) *Suppose that in the above situation* $E = \operatorname{curl} F$ *for some vector field* F. *What conditions, if any, does this place on* a *and* b*?*

[Answer: $a = b = 0$*]*

Problem 43 (Stewart, Exercise 16.9.31). *Suppose that* S *and* E *satisfy the conditions of the Divergence Theorem and* f *is a scalar function having second-partial derivatives. Prove that*

$$\iint_S fn\, dS = \iiint_E \nabla f\, dV.$$

These surface and triple integrals of vector functions are vectors defined by integrating each component function. [Hint: Start by applying the Divergence Theorem to $F = fc$*, where* c *is an arbitrary constant vector.]*

Problem 44 (Cal Final, Summer 2018W). *Let* C *be the curve of intersection of the cylinder* $x^2 + y^2 = 1$ *and the plane* $z = 2x + 3y$*, oriented counterclockwise when viewed from above. Let*

$$F = \langle x^{2018} + y, y^{2018} + z, z^{2018} + x\rangle.$$

Calculate $\int_C F \cdot dr$.

Problem 45 (Cal Final, Summer 2018W). *Calculate the flux* $\iint_S F \cdot dS$*, where* S *is the hemisphere* $x^2 + y^2 + z^2 = 1$ *with* $z \geq 0$ *oriented upwards, and* $F = \langle x + \sin y, y + \cos z, z + 1\rangle$.

References

[1] J. Stewart: *Calculus* 8th Edition, Cengage Learning, Boston 2016.