VECTOR CALCULUS (MATH 53)

1. Parametric Curves

TODO: add to surfaces problem: Is the surface $x^2 + 4xy + 3yz + z^2 = 0x2 + 4xy + 3yz + z2 = 0$ some kind of cone (a union of lines through the origin)?

Problem 1 (Cal Final, Fall 09)**.** *Consider the parameterized curve*

$$x = te^{-t}, \qquad y = e^{3t}.$$

(1) What is the equation of the tangent line of the curve at the point $(0, 1)$?
(2) Find all points (x, y) on the curve at which the curve is vertical.

Solution: (1) First let us compute the slope:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3e^{3t}}{e^{-t}(1-t)} = \frac{3e^{4t}}{1-t}.$$

At $x = 0$, we have that $t = 0$. Therefore $(dy/dx)(0) = 3$, which implies that the equation of the line is $y = 3x + 1$.

(2) The curve is vertical when $dx/dt = 0$ and $dy/dt \neq 0$. Then $e^{-t}(1-t) = 0$, which implies $t = 1$. As $(dy/dt)(1) \neq 0$, it follows that the only point where the curve is vertical is (e^{-1}, e^3) when $t = 1$. □

Problem 2. *Find the length of the curve*

$$x = e^t + e^{-t}, \qquad y = 2018 + 2t.$$

Solution: Since $dx/dt = e^t - e^{-t}$ and $dy/dt = 2$, the length of the curve is given by

$$\int_0^1 \sqrt{(e^t - e^{-t})^2 + 4}\, dt = \int_0^1 \sqrt{e^{2t} + 2 + e^{-2t}}\, dt = \int_0^1 (e^t + e^{-t})\, dt = e - e^{-1}.$$

□

2. Polar Coordinates

Problem 3. *Suppose that the polar curve $r = f(\theta)$, where $0 \le \theta \le 2\pi$ has length L and encloses a region of area A.*

(1) What is the area of the region enclosed by the polar curve $r = 4f(\theta)$, where $0 \le \theta \le 2\pi$?
(2) What is the length of the polar curve $r = 5f(\theta)$, where $0 \le \theta \le 2\pi$?

Solution: Let A' be the area of the region enclosed by the polar curve $r = 4f(\theta)$, where $0 \le \theta \le 2\pi$, and let L' be the length of the polar curve $r = 5f(\theta)$, where $0 \le \theta \le 2\pi$. Then

$$A' = \int_0^{2\pi} \frac{1}{2}[4f(\theta)]^2 \, d\theta = 16A \tag{1}$$

and

$$L' = \int_0^{2\pi} \sqrt{[5f(\theta)]^2 + [\frac{d(5f(\theta))}{d\theta}]^2} \, d\theta = 5 \int_0^{2\pi} \sqrt{[f(\theta)]^2 + [\frac{d(f(\theta))}{d\theta}]^2} \, d\theta = 5L. \tag{2}$$

$\square$

Problem 4. *(1) Sketch the region that lies inside both of the polar curves $r = 1 + \cos\theta$ and $r = 1 - \cos\theta$.*
(2) Compute the area of the region.

Solution:
(1) The red part of the figure shows the desired enclosed region:

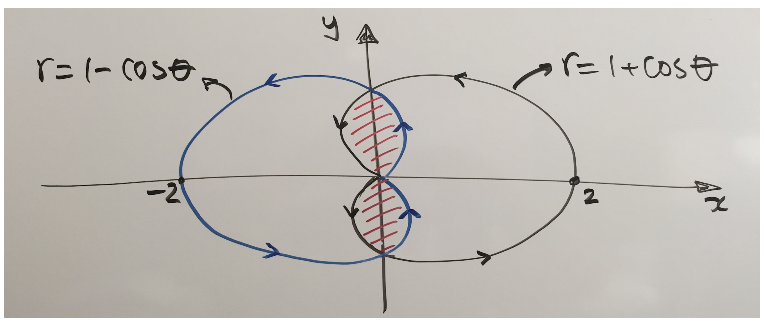

FIGURE 1. Colored red, the region between $r = 1 + \cos\theta$ and $r = 1 - \cos\theta$.

(2) By symmetry, we that the area enclosed by the the two curves is

$$4 \int_0^{\pi/2} \frac{1}{2}\left(1 - \cos\theta\right)^2 d\theta = \frac{3\pi}{2} - 4.$$

$\square$

REFERENCES

[1] J. Stewart: *Calculus* 8th Edition, Cengage Learning, Boston 2016.