VECTOR CALCULUS (MATH 53)

1. Double and Triple Integrals

Problem 1 (Stewart, Exercise 15.1.11). *Evaluate the double integral*

$$\iint_R (4-2y)\,dA, \quad R=[0,2]\times[0,1],$$

by first identifying it as the volume of a solid.

Problem 2 (Stewart, Exercise 15.1.34). *Evaluate the double integral*

$$\iint_R \frac{1}{1+x+y}\,dA, \quad R=[1,3]\times[1,2].$$

Problem 3 (Stewart, Exercise 15.1.42). *Find the volume of the solid in the first octant bounded by the cylinder $z=16-x^2$ and the plane $y=5$.*

Problem 4 (Stewart, Exercise 15.1.49). *Use symmetry to evaluate the double integral*

$$\iint_R \frac{xy}{1+x^4}\,dA, \quad R=[-1,1]\times[0,1].$$

Problem 5 (Stewart, Exercise 15.2.22). *Evaluate the double integral*

$$\iint_D y\,dA,$$

where D is the triangular region with vertices $(0,0)$, $(1,1)$, and $(4,0)$.

Problem 6 (Stewart, Exercise 15.2.27). *Find the volume of the tetrahedron enclosed by the coordinate planes and the plane $2x+y+z=4$.*

Problem 7 (Stewart, Example 15.2.4). *Find the volume of the tetrahedron bounded by the planes $x+2y+z=2$, $x=2y$, and $z=0$ using:*

(1) double integrals,
(2) triple integrals.

[Hint: The tetrahedron lies in the first octant. Sketch it by looking from your left.]

Problem 8 (Stewart, Exercise 15.2.40). *Sketch the solid whose volume is given by the integral*

$$\int_0^1 \int_0^{1-x^2} (1-x)\,dy dx.$$

Problem 9 (Stewart, Exercise 15.2.(51,52,55)). *Evaluate the following integrals:*

(1) $\int_0^1 \int_{3y}^3 e^{x^2}\, dxdy$,

(2) $\int_0^1 \int_{x^2}^1 \sqrt{y}\sin y\, dydx$,

(3) $\int_0^1 \int_{\arcsin y}^{\pi/2} \cos x\sqrt{1+\cos^2 x}\, dxdy$.

[Hint: Reverse the order of integration first.]

Problem 10 (Cal Final, Summer 2018W). *Calculate the iterated integral*

$$\int_0^4 \int_{\sqrt{x}}^2 e^{y^3}\, dydx.$$

Problem 11 (Stewart, Exercise 15.2.(65,67,68)). *Use geometry to evaluate the following double integrals.*

(1) $\iint_D (x+2)\, dA, \quad D = \{(x,y) \mid 0 \le y \le \sqrt{9-x^2}\}$,

(2) $\iint_D (2x+3y)\, dA, \quad D = [0,a]\times[0,b]$,

(3) $\iint_D (2 + x^2y^3 - y^2\sin x)\, dA, \quad D = \{(x,y) \mid |x|+|y| \le 1\}$.

Problem 12 (Stewart, Exercise 15.4.9). *Find the mass and center of mass of the lamina that occupies the region bounded by the curves $y = e^{-x}$, $y = 0$, $x = 0$, and $x = 1$ if its density function is $\rho(x,y) = xy$.*

Problem 13 (Stewart, Exercise 15.5.3). *Use the formula*

$$A(S) = \iint_D \sqrt{1 + f_x^2 + f_y^2}\, dA$$

to find the surface area of the part of the plane $3x + 2y + z = 6$ that lies in the first octant.

Problem 14 (Stewart, Exercise 15.5.7). *Find the surface area of the part of the hyperbolic paraboloid $z = y^2 - x^2$ that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$. [Hint: Use the formula for surface area via double integrals.]*

Problem 15 (Stewart, Exercise 15.6.29). *Express the integral $\iiint_E f(x,y,z)\, dV$ as an iterated integral in six different ways, where E is the solid bounded by $y = 4 - x^2 - 4z^2$ and $y = 0$.*

Problem 16 (Stewart, Exercise 15.6.33). *Sketch the solid whose volume is given by the triple integral*

$$\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x,y,z)\, dzdydx.$$

Write five other iterated integrals that are equal to the given iterated integral. [Note: Sketch given in the book.]

Problem 17 (Stewart, Exercise 15.6.36). *Write five other iterated integrals that are equal to the iterated integral*

$$\int_0^1 \int_y^1 \int_0^z f(x,y,z)\,dxdzdy.$$

Problem 18 (Stewart, Exercise 15.6.(37,38)). *Use geometry to evaluate the following double integrals.*

(1) $\iiint_C (4+5x^2yz^2)\,dV$, *where* C *is the cylindrical region given by* $x^2+y^2 \leq 4$ *and* $-2 \leq z \leq 2$.

(2) $\iiint_B (z^3+\sin y+3)\,dV$, *where* B *is the unit ball* $x^2+y^2+z^2 \leq 1$.

Problem 19 (Stewart, Exercise 15.6.42). *Find the mass and center of mass of the tetrahedron in the first octant bounded by the plane* $x+y+z=1$ *with density given by* $\rho(x,y,z)=y$.

2. Polar, Cylindrical, and Spherical Coordinates

Problem 20 (Stewart, Exercise 15.3.(15,17)). *Use double integrals to find the areas of the following regions.*

(1) *One loop of the rose* $r=\cos 3\theta$.

(2) *The region inside the circle* $(x-1)^2+y^2=1$ *and outside the circle* $x^2+y^2=1$.

Problem 21 (Stewart, Exercise 15.3.27). *Use polar coordinate to find the volume of the solid inside both the cylinders* $x^2+y^2=4$ *and the ellipsoid* $4x^2+4y^2+z^2=64$.

Problem 22 (Stewart, Exercise 15.3.(29,30)). *Evaluate the following integrals:*

$$(1)\quad \int_0^2 \int_0^{\sqrt{4-x^2}} e^{-x^2-y^2}\,dydx \qquad\qquad (2)\quad \int_0^a \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} (2x+y)\,dxdy.$$

Problem 23 (Stewart, Exercise 15.3.35). *A swimming pool is circular with a 40-feet diameter. The depth is constant along east-west lines and increases linearly from 2 feet at the south end to 7 feet at the north end. Find the volume of water in the pool.*

Problem 24 (Stewart, Exercise 15.7.(15,16)). *In each case, sketch the solid whose volume is given by the following integrals:*

(1) $\int_{-\pi/2}^{\pi/2} \int_0^2 \int_0^{r^2} r\,dzdrd\theta$,

(2) $\int_0^2 \int_0^{2\pi} \int_0^r r\,dzd\theta dr$.

Problem 25 (Stewart, Exercise 15.7.21). *Evaluate* $\iiint_E x^2\,dV$, *where* E *is the solid that lies within the cylinder* $x^2+y^2=1$, *above the plane* $z=0$, *and below the cone* $z^2=4x^2+4y^2$.

Problem 26 (Stewart, Exercise 15.7.25). *Find the volume of the region E that lies between the paraboloid $z = 24 - x^2 - y^2$ and the cone $z = 2\sqrt{x^2+y^2}$. Find the centroid of E (i.e., the center of mass in constant density).*

Problem 27 (Stewart, Exercise 15.7.29). *Evaluate*

$$\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{\sqrt{x^2+y^2}}^{2} xz\, dzdxdy.$$

[Hint: Use cylindrical coordinates.]

Problem 28 (Stewart, Exercise 15.8.(17,18)). *In each case, sketch the solid whose volume is given by the following integrals:*

(1) $\int_0^{\pi/6}\int_0^{\pi/2}\int_0^3 \rho^2\sin\phi\, d\rho d\theta d\phi$,

(2) $\int_0^{\pi/4}\int_0^{2\pi}\int_0^{\sec\phi} \rho^2\sin\phi\, d\rho d\theta d\phi$.

Problem 29 (Stewart, Exercise 15.8.27). *Find the volume of the part of the ball $\rho \le a$ that lies between the cones $\phi = \pi/6$ and $\phi = \pi/3$.*

Problem 30 (Stewart, Exercise 15.8.28). *Find the average distance from a point inside a ball of radius a to its center.*

Problem 31 (Stewart, Exercise 15.8.30). *Find the volume of the solid that lies within the sphere $x^2+y^2+z^2 = 4$, above the xy-plane, and below the cone $z = \sqrt{x^2+y^2}$.*

Problem 32 (Stewart, Exercise 15.8.41). *Evaluate*

$$\int_0^1\int_0^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} xy\, dzdydx.$$

[Hint: Use spherical coordinates.]

Problem 33 (Cal Final, Summer 2018W). *Let E be the region defined by the inequalities*

$$x^2+y^2+z^2 \le 4, \quad 0 \le y \le x, \quad z \ge 0.$$

Calculate the total mass of E if the mass density is given by z^2.

3. Change of Coordinates

Problem 34 (Stewart, Example 15.9.(23,25,27)). *Evaluate the following integrals.*

(1) $\iint_R \frac{x-2y}{3x-y}\, dA$, *where R is the parallelogram enclosed by the lines $x - 2y = 0$, $x - 2y = 4$, $3x - y = 1$, and $3x - y = 8$.*

(2) $\iint_R \cos\left(\frac{y-x}{y+x}\right) dA$, *where R is the trapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,2)$, and $0,1$.*

(3) $\iint_R e^{x+y}\,dA$, *where* R *is given by the inequality* $|x| + |y| \leq 1$.

Problem 35 (Stewart, Example 15.9.3)**.** *Evaluate* $\int\int_R e^{(x+y)/(x-y)} d\,A$, *where* R *is the trapezoidal region with vertices* $(1, 0)$, $(2, 0)$, $(0, -2)$, *and* $(0, -1)$. *[Hint: Define* T *by* $u = x + y$ *and* $v = x - y$ *and find that* T^{-1} *is defined by* $x = 1/2(u + v)$ *and* $y = 1/2(u - v)$ *and, therefore,* $\partial(x, y)/\partial(u, v) = -1/2$*][Answer:* $3/4(e - e^{-1})$*]*

Problem 36 (Stewart, Exercise 5.9.28)**.** *Let* f *be continuous on* $[0, 1]$ *and let* R *be the triangular region with vertices* $(0, 0)$, $(1, 0)$, *and* $(0, 1)$. *Show that*

$$\iint_R f(x + y)\,dA = \int_0^1 uf(u)\,du.$$

Problem 37 (Stewart, Exercise 5.56)**.** *Find the volume of the region bounded by the surface* $\sqrt{x} + \sqrt{y} + \sqrt{z} = 1$ *and the coordinate planes. [Hint: Use the change of coordinates* $x = u^2$, $y = v^2$, *and* $z = w^2$.*]*

Problem 38 (Stewart, Exercise 5.57)**.** *Evaluate* $\iint_R xy\,dA$, *where* R *is the square with vertices* $(0, 0)$, $(1, 1)$, $(2, 0)$, *and* $(1, -1)$. *[Hint: Use an appropriate change of coordinates.]*

Problem 39 (Cal Final, Summer 2018W)**.** *Let* D *be the region in the plane where* $x^2 + 2y^2 \leq 1$. *Compute the double integral*

$$\iint_D (x^2 + 2y^2)^{2018}\,dA.$$

[Hint: Use $u = x$ *and* $y = \sqrt{2}y$. *Answer:* $\sqrt{2}\pi/4038$.*]*

References

[1] J. Stewart: *Calculus* 8th Edition, Cengage Learning, Boston 2016.