VECTOR CALCULUS (MATH 53)

1. Double and Triple Integrals

Problem 1 (Stewart, Exercise 15.1.11). *Evaluate the double integral*

$$\iint_R (4 - 2y)\, dA, \quad R = [0,2] \times [0,1],$$

by first identifying it as the volume of a solid.

Problem 2 (Stewart, Exercise 15.1.34). *Evaluate the double integral*

$$\iint_R \frac{1}{1 + x + y}\, dA, \quad R = [1,3] \times [1,2].$$

Problem 3 (Stewart, Exercise 15.1.42). *Find the volume of the solid in the first octant bounded by the cylinder $z = 16 - x^2$ and the plane $y = 5$.*

Problem 4 (Stewart, Exercise 15.1.49). *Use symmetry to evaluate the double integral*

$$\iint_R \frac{xy}{1 + x^4}\, dA, \quad R = [-1,1] \times [0,1].$$

Problem 5 (Stewart, Exercise 15.2.22). *Evaluate the double integral*

$$\iint_D y\, dA,$$

where D is the triangular region with vertices $(0,0)$, $(1,1)$, and $(4,0)$.

Problem 6 (Stewart, Exercise 15.2.27). *Find the volume of the tetrahedron enclosed by the coordinate planes and the plane $2x + y + z = 4$.*

Problem 7 (Stewart, Example 15.2.4). *Find the volume of the tetrahedron bounded by the planes $x + 2y + z = 2$, $x = 2y$, and $z = 0$ using:*

    (1) double integrals,
    (2) triple integrals.

[Hint: The tetrahedron lies in the first octant. Sketch it by looking from your left.]

Problem 8 (Stewart, Exercise 15.2.40). *Sketch the solid whose volume is given by the integral*

$$\int_0^1 \int_0^{1-x^2} (1 - x)\, dydx.$$

Problem 9 (Stewart, Exercise 15.2.(51,52,55)). *Evaluate the following integrals:*

(1) $\int_0^1 \int_{3y}^3 e^{x^2} \, dx dy$,

(2) $\int_0^1 \int_{x^2}^1 \sqrt{y} \sin y \, dy dx$,

(3) $\int_0^1 \int_{\arcsin y}^{\pi/2} \cos x \sqrt{1 + \cos^2 x} \, dx dy$.

[Hint: Reverse the order of integration first.]

Problem 10 (Cal Final, Summer 2018W). *Calculate the iterated integral*

$$\int_0^4 \int_{\sqrt{x}}^2 e^{y^3} \, dy dx.$$

Problem 11 (Stewart, Exercise 15.2.(65,67,68)). *Use geometry to evaluate the following double integrals.*

(1) $\iint_D (x+2) \, dA$, $\quad D = \{(x,y) \mid 0 \le y \le \sqrt{9 - x^2}\}$,

(2) $\iint_D (2x + 3y) \, dA$, $\quad D = [0,a] \times [0,b]$,

(3) $\iint_D (2 + x^2 y^3 - y^2 \sin x) \, dA$, $\quad D = \{(x,y) \mid |x| + |y| \le 1\}$.

Problem 12 (Stewart, Exercise 15.4.9). *Find the mass and center of mass of the lamina that occupies the region bounded by the curves $y = e^{-x}$, $y = 0$, $x = 0$, and $x = 1$ if its density function is $\rho(x,y) = xy$.*

Problem 13 (Stewart, Exercise 15.5.3). *Use the formula*

$$A(S) = \iint_D \sqrt{1 + f_x^2 + f_y^2} \, dA$$

to find the surface area of the part of the plane $3x + 2y + z = 6$ that lies in the first octant.

Problem 14 (Stewart, Exercise 15.5.7). *Find the surface area of the part of the hyperbolic paraboloid $z = y^2 - x^2$ that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$. [Hint: Use the formula for surface area via double integrals.]*

Problem 15 (Stewart, Exercise 15.6.29). *Express the integral $\iiint_E f(x,y,z) \, dV$ as an iterated integral in six different ways, where E is the solid bounded by $y = 4 - x^2 - 4z^2$ and $y = 0$.*

Problem 16 (Stewart, Exercise 15.6.33). *Sketch the solid whose volume is given by the triple integral*

$$\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x,y,z) \, dz dy dx.$$

Write five other iterated integrals that are equal to the given iterated integral. [Note: Sketch given in the book.]

Problem 17 (Stewart, Exercise 15.6.36)**.** *Write five other iterated integrals that are equal to the iterated integral*

$$\int_0^1 \int_y^1 \int_0^z f(x, y, z)\, dx dz dy.$$

Problem 18 (Stewart, Exercise 15.6.(37,38))**.** *Use geometry to evaluate the following double integrals.*

(1) $\iiint_C (4 + 5x^2 yz^2)\, dV$, *where* C *is the cylindrical region given by* $x^2 + y^2 \leq 4$ *and* $-2 \leq z \leq 2$.

(2) $\iiint_B (z^3 + \sin y + 3)\, dV$, *where* B *is the unit ball* $x^2 + y^2 + z^2 \leq 1$.

Problem 19 (Stewart, Exercise 15.6.42)**.** *Find the mass and center of mass of the tetrahedron in the first octant bounded by the plane* $x + y + z = 1$ *with density given by* $\rho(x, y, z) = y$.

2. Polar, Cylindrical, and Spherical Coordinates

Problem 20 (Stewart, Exercise 15.3.(15,17))**.** *Use double integrals to find the areas of the following regions.*

(1) *One loop of the rose* $r = \cos 3\theta$.

(2) *The region inside the circle* $(x-1)^2 + y^2 = 1$ *and outside the circle* $x^2 + y^2 = 1$.

Problem 21 (Stewart, Exercise 15.3.27)**.** *Use polar coordinate to find the volume of the solid inside both the cylinders* $x^2 + y^2 = 4$ *and the ellipsoid* $4x^2 + 4y^2 + z^2 = 64$.

Problem 22 (Stewart, Exercise 15.3.(29,30))**.** *Evaluate the following integrals:*

(1) $\displaystyle \int_0^2 \int_0^{\sqrt{4-x^2}} e^{-x^2 - y^2}\, dy dx$

(2) $\displaystyle \int_0^a \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} (2x + y)\, dx dy.$

Problem 23 (Stewart, Exercise 15.3.35)**.** *A swimming pool is circular with a 40-feet diameter. The depth is constant along east-west lines and increases linearly from 2 feet at the south end to 7 feet at the north end. Find the volume of water in the pool.*

Problem 24 (Stewart, Exercise 15.7.(15,16))**.** *In each case, sketch the solid whose volume is given by the following integrals:*

(1) $\int_{-\pi/2}^{\pi/2} \int_0^2 \int_0^{r^2} r\, dz dr d\theta,$

(2) $\int_0^2 \int_0^{2\pi} \int_0^r r\, dz d\theta dr.$

Problem 25 (Stewart, Exercise 15.7.21)**.** *Evaluate* $\iiint_E x^2\, dV$, *where* E *is the solid that lies within the cylinder* $x^2 + y^2 = 1$, *above the plane* $z = 0$, *and below the cone* $z^2 = 4x^2 + 4y^2$.

Problem 26 (Stewart, Exercise 15.7.25). *Find the volume of the region E that lies between the paraboloid $z = 24 - x^2 - y^2$ and the cone $z = 2\sqrt{x^2 + y^2}$. Find the centroid of E (i.e., the center of mass in constant density).*

Problem 27 (Stewart, Exercise 15.7.29). *Evaluate*

$$\int_{-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_{\sqrt{x^2+y^2}}^{2} xz \, dzdxdy.$$

[Hint: Use cylindrical coordinates.]

Problem 28 (Stewart, Exercise 15.8.(17,18)). *In each case, sketch the solid whose volume is given by the following integrals:*

(1) $\int_0^{\pi/6} \int_0^{\pi/2} \int_0^3 \rho^2 \sin \phi \, d\rho d\theta d\phi$,

(2) $\int_0^{\pi/4} \int_0^{2\pi} \int_0^{\sec \phi} \rho^2 \sin \phi \, d\rho d\theta d\phi.$

Problem 29 (Stewart, Exercise 15.8.27). *Find the volume of the part of the ball $\rho \leq a$ that lies between the cones $\phi = \pi/6$ and $\phi = \pi/3$.*

Problem 30 (Stewart, Exercise 15.8.28). *Find the average distance from a point inside a ball of radius a to its center.*

Problem 31 (Stewart, Exercise 15.8.30). *Find the volume of the solid that lies within the sphere $x^2 + y^2 + z^2 = 4$, above the xy-plane, and below the cone $z = \sqrt{x^2 + y^2}$.*

Problem 32 (Stewart, Exercise 15.8.41). *Evaluate*

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} xy \, dzdydx.$$

[Hint: Use spherical coordinates.]

Problem 33 (Cal Final, Summer 2018W). *Let E be the region defined by the inequalities*

$$x^2 + y^2 + z^2 \leq 4, \quad 0 \leq y \leq x, \quad z \geq 0.$$

Calculate the total mass of E if the mass density is given by z^2.

3. Change of Coordinates

Problem 34 (Stewart, Example 15.9.(23,25,27)). *Evaluate the following integrals.*

(1) $\iint_R \frac{x-2y}{3x-y} \, dA$, *where R is the parallelogram enclosed by the lines $x - 2y = 0$, $x - 2y = 4$, $3x - y = 1$, and $3x - y = 8$.*

(2) $\iint_R \cos \left(\frac{y-x}{y+x} \right) dA$, *where R is the trapezoidal region with vertices $(1, 0)$, $(2, 0)$, $(0, 2)$, and $0, 1$.*

(3) $\iint_R e^{x+y} \, dA$, where R is given by the inequality $|x| + |y| \le 1$.

Problem 35 (Stewart, Example 15.9.3)**.** *Evaluate* $\int\int_R e^{(x+y)/(x-y)} dA$, *where R is the trapezoidal region with vertices* $(1,0)$, $(2,0)$, $(0,-2)$, *and* $(0,-1)$. *[Hint: Define T by $u = x + y$ and $v = x - y$ and find that T^{-1} is defined by $x = 1/2(u+v)$ and $y = 1/2(u-v)$ and, therefore, $\partial(x,y)/\partial(u,v) = -1/2$][Answer: $3/4(e - e^{-1})$]*

Problem 36 (Stewart, Exercise 5.9.28)**.** *Let f be continuous on $[0,1]$ and let R be the triangular region with vertices $(0,0)$, $(1,0)$, and $(0,1)$. Show that*

$$\iint_R f(x+y) \, dA = \int_0^1 u f(u) \, du.$$

Problem 37 (Stewart, Exercise 5.56)**.** *Find the volume of the region bounded by the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = 1$ and the coordinate planes. [Hint: Use the change of coordinates $x = u^2$, $y = v^2$, and $z = w^2$.]*

Problem 38 (Stewart, Exercise 5.57)**.** *Evaluate $\iint_R xy \, dA$, where R is the square with vertices $(0,0)$, $(1,1)$, $(2,0)$, and $(1,-1)$. [Hint: Use an appropriate change of coordinates.]*

Problem 39 (Cal Final, Summer 2018W)**.** *Let D be the region in the plane where $x^2 + 2y^2 \le 1$. Compute the double integral*

$$\iint_D (x^2 + 2y^2)^{2018} \, dA.$$

[Hint: Use $u = x$ and $y = \sqrt{2}y$. Answer: $\sqrt{2}\pi/4038$.]

REFERENCES

[1] J. Stewart: *Calculus* 8th Edition, Cengage Learning, Boston 2016.