SOLUTIONS FOR QUIZ 2

Note: Most of the problems were taken from the textbook [1].

Problem 1. *Determine whether the sequence* $a_n = \cos\left(\sqrt[n]{\pi^{207+n}}\right)$ *converges or diverges. If it is convergent, find the limit.*

Solution: Since $\cos x$ and π^x are both continuous functions we have that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \cos\left(\pi^{\frac{207}{n}+1}\right) = \cos\left(\pi^{\lim \frac{207}{n}+1}\right) = \cos(\pi) = -1.$$

Hence $\{a_n\}$ is a convergent sequence. □

Problem 2. *Determine whether the series*

$$\sum_{n=1}^{\infty} \sqrt[3]{\frac{n^2}{n^2+20n+9}}$$

is convergent or divergent. If it is convergent, find the sum.

Solution: As $\sqrt[3]{x}$ is a continuous function, it follows that

$$\lim_{n\to\infty} \sqrt[3]{\frac{n^2}{n^2+20n+9}} = \sqrt[3]{\lim_{n\to\infty} \frac{1}{1+20/n+9/n^2}} = \sqrt[3]{1} = 1.$$

Since the general term of the series does not converges to zero, the series diverges. □

Problem 3. *Find the exact area of the surface of revolution obtained by rotating the curve* $x = 1+2y^2,\ 1 \le y \le 2$ *about the* x*-axis.*

Solution: From $x = 1+2y^2$, we find that $f(x) = y = \sqrt{\frac{x-1}{2}}$. Since $1 \le y \le 2$, one has that $3 \le x \le 9$. In addition,

$$1 + [f(x)']^2 = 1 + \left(\frac{1}{\sqrt{2}}\frac{1}{2\sqrt{x-1}}\right)^2 = 1 + \frac{1}{8(x-1)} = \frac{8x-7}{8(x-1)}.$$

Therefore the area of the surface of revolution is

$$2\pi \int_3^9 \sqrt{\frac{x-1}{2}}\sqrt{\frac{8x-7}{8(x-1)}}\,dx = \frac{\pi}{2}\int_3^9 (8x-7)^{1/2}\,dx$$
$$= \frac{\pi}{16}\int_{17}^{65} u^{1/2}\,du = \frac{\pi}{24}\left(\sqrt{65^3} - \sqrt{17^3}\right).$$

□

Problem 4. *Use the Integral Test to determine whether the series* $\sum_{n=1}^{\infty} n^2\pi^{-n^3}$ *is convergent or divergent.*

Solution: The function $f(x) = x^2\pi^{-x^3}$ is continuous and positive for all $x \geq 1$. Moreover,

$$f(x)' = \frac{\pi^{x^3}\left(2x - (3\ln\pi)x^4\right)}{\pi^{2x^3}} < 0$$

for all $x \geq 1$. Then $f(x)$ is decreasing when $x \geq 1$. Taking $u = x^3$ below, we find that

$$\int_1^{\infty} f(x)\,dx = \frac{1}{3}\int_1^{\infty}\left(\frac{1}{\pi}\right)^u du = \frac{1}{3\ln(1/\pi)}(-1/\pi) = \frac{1}{3\pi\ln\pi}.$$

Since the above improper integral is finite, the series converges by the Integral Test. □

Problem 5. *Determine whether the series*

$$\sum_{n=1}^{\infty}\left(\frac{2}{(n+1)(n+3)} + \frac{3}{\pi^{n+1}}\right)$$

is convergent or divergent. If it is convergent, find the sum.

Solution: Notice that

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\frac{2}{(n+1)(n+3)} + \frac{3}{\pi^{n+1}}\right) &= \sum_{n=1}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+3}\right) + \frac{3}{\pi^2}\sum_{n=0}^{\infty}\left(\frac{1}{\pi}\right)^n \\ &= \lim_{n\to\infty}\left(\frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3}\right) + \frac{3/\pi^2}{1-1/\pi} \\ &= \frac{5}{6} + \frac{3}{\pi^2 - \pi}.\end{aligned}$$

Hence the series converges to $5/6 + 3/(\pi^2 - \pi)$. □

References

[1] J. Stewart: *Single Variable Calculus* 8th Edition, Cengage Learning, Boston 2015.