MOCK QUIZ 2

Note: Most of the problems were taken from the textbook [1].

Problem 1. *If $a, b \in \mathbb{R}$ such that $a, b > 0$, compute the limit of the sequence $\{a_n\}$, where*

$$a_n = \left(1 + \frac{a}{n}\right)^{\frac{n}{b}}.$$

Solution 1: We can compute the limit of $\{a_n\}$ as follows:

$$\begin{aligned}\lim_{n\to\infty} a_n = \lim_{n\to\infty} \exp\left(\frac{n}{b}\ln\left(1+\frac{a}{n}\right)\right) &= \exp\left(\lim_{n\to\infty}\frac{n}{b}\ln\left(1+\frac{a}{n}\right)\right)\\ = \exp\left(\lim_{x\to\infty}\frac{\ln\left(\frac{x+a}{x}\right)}{\frac{b}{x}}\right) &= \exp\left(\lim_{x\to\infty}\frac{\frac{x}{x+a}\frac{-a}{x^2}}{\frac{-b}{x^2}}\right)\\ = \exp\left(\lim_{x\to\infty}\frac{a}{b}\frac{x}{x+a}\right) &= e^{\frac{a}{b}}.\end{aligned}$$

Solution 2: We can compute the limit of $\{a_n\}$ as follows:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1+\frac{a}{n}\right)^{\frac{n}{b}} = \lim_{n\to\infty}\left(\left(1+\frac{a}{n}\right)^{\frac{n}{a}}\right)^{\frac{a}{b}} = \left(\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^{\frac{n}{a}}\right)^{\frac{a}{b}} = e^{\frac{a}{b}}.$$

$\square$

Problem 2. *Decide whether the series*

$$\sum_{n=2}^{\infty}\ln\left(\frac{n+1}{n}\right) \tag{0.1}$$

converges. If it is convergent, then find the limit.

Solution: Denoting the n-th partial sum of (0.1) by s_n, we can observe that

$$s_n = \sum_{k=2}^{n}\ln\left(\frac{k+1}{k}\right) = \sum_{k=2}^{n}\big(\ln(k+1) - \ln k\big) = \ln(n+1) - \ln 2.$$

Therefore

$$\sum_{n=2}^{\infty}\ln\left(\frac{n+1}{n}\right) = \lim_{n\to\infty}\ln(n+1) - \ln 2 = \infty.$$

$\square$

References

[1] J. Stewart: *Single Variable Calculus* 8th Edition, Cengage Learning, Boston 2015.