MATH1B: FINAL PRACTICE EXAM **TIME: 2 HOURS**

Problem 1. *Evaluate the following integrals*

$$(a) \int \frac{dy}{16y + 10y\ln y + y(\ln y)^2} \qquad (b) \int e^{-x}\cos 2x\,dx.$$

Solution: (a) Taking $x = \ln y$ and, therefore, $dx = dy/y$, we get

$$\begin{aligned}\int \frac{dy}{16y + 10y\ln y + y(\ln y)^2} &= \int \frac{dx}{16 + 10x + x^2} = \int \frac{dx}{(x+2)(x+8)} \\ &= \frac{1}{6}\left(\int \frac{dx}{x+2} - \frac{dx}{x+8}\right) = \frac{1}{6}\ln|x+2| - \frac{1}{6}\ln|x+8| + C \\ &= \frac{1}{6}\ln|2+\ln y| - \frac{1}{6}\ln|8+\ln y| + C.\end{aligned}$$

(b) Integrating by parts taking $u = e^{-x}$ and $dv = \cos 2x dx$, one obtains

$$\int e^{-x}\cos 2x\,dx = \frac{1}{2}e^{-x}\sin 2x + \frac{1}{2}\int e^{-x}\sin 2x\,dx. \tag{0.1}$$

Using integration by parts on the integral in the right-hand side of (0.1) with $u = e^{-x}$ and $dv = \sin 2x\,dx$, we have that

$$\int e^{-x}\cos 2x\,dx = \frac{1}{2}e^{-x}\sin 2x + \frac{1}{2}\left(-\frac{1}{2}e^{-x}\cos 2x - \frac{1}{2}\int e^{-x}\cos 2x\,dx\right).$$

Hence, by factoring the integrals in the above equations, we conclude that

$$\int e^{-x}\cos 2x\,dx = \frac{2}{5}e^{-x}\sin 2x - \frac{1}{5}e^{-x}\cos 2x.$$

□

Problem 2. *Decide whether the next integral is divergent or convergent. If it is convergent, evaluate it:*

$$\int_0^{\pi/2} \frac{\cos\theta}{\sqrt{\sin\theta}}\,d\theta.$$

Solution: Using the definition of improper integral and then the substitution $u = \sin\theta$, it follows that

$$\int_0^{\pi/2} \frac{\cos\theta}{\sqrt{\sin\theta}}\,d\theta = \lim_{t\to 0^+}\int_t^{\pi/2} \frac{\cos\theta}{\sqrt{\sin\theta}}\,d\theta = \lim_{t\to 0^+}\int_{\sin t}^{1} \frac{du}{\sqrt{u}} = \lim_{t\to 0^+} 2\sqrt{u}\Big|_{\sin t}^{1} = 2.$$

Therefore the integral is convergent and its limit is 2. □

Problem 3. *For the function $f(x) = \frac{1}{4}e^x + e^{-x}$, argue that the arc length on any interval has the same value as the area under the curve.*

Solution: First, notice that

$$\sqrt{1 + \left(f'(x)\right)^2} = \sqrt{1 + \left(\frac{e^{2x}}{16} - \frac{1}{2} + e^{-2x}\right)} = \sqrt{\left(\frac{e^x}{4} + e^{-x}\right)^2} = f(x).$$

Thus, if $a < b$ and $L_f([a, b])$ is the arc length of $f(x)$ when $a \le x \le b$, then

$$L_f([a, b]) = \int_a^b \sqrt{1 + \left(f'(x)\right)^2}\, dx = \int_a^b f(x)\, dx,$$

which is the area under curve $f(x)$ from a to b. □

Problem 4. *Determine whether the following series is absolutely convergent, conditionally convergent, or divergent:*

$$\sum_{n=1}^{\infty} (-1)^n \frac{\arctan n}{n^2}.$$

Solution: Since

$$\sum_{n=1}^{\infty} \left| (-1)^n \frac{\arctan n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{|\arctan n|}{n^2} \le \sum_{n=1}^{\infty} \frac{\pi/2}{n^2},$$

which converges, by the Comparison Test the given series is absolutely convergent. □

Problem 5. *Find the radius of convergence and the interval of convergence of the following power series:*

$$(a)\ \sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n \sqrt{n}} \qquad (b)\ \sum_{n=2}^{\infty} \frac{b^n}{\ln n}(x-a)^n, \quad b > 0.$$

Solution: (a) Since

$$\lim_{n\to\infty} \frac{|2x-1|^{n+1}}{5^{n+1}\sqrt{n+1}} \cdot \frac{5^n \sqrt{n}}{|2x-1|^n} = |2x-1| \lim_{n\to\infty} \frac{1}{5}\sqrt{\frac{n}{n+1}} = \frac{|2x-1|}{5},$$

it follows by the Ratio Test that radius of convergence is $5/2$. Substituting $2x - 1 = -5$ and $2x - 1 = 5$ in the original power series, we can see that

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

are convergent (by the Alternating Test) and divergent (by the p-test), respectively. Hence the interval of convergence is $[-2, 3)$.

(b) The solution is similar to part (a): the radius of convergence is $1/b$ and the interval of convergence is $[a - 1/b, a + 1/b)$. □

Problem 6. *Find the Taylor series of the following functions:*

$$(a)\ f(x) = \frac{x^2}{\sqrt{2+x}} \qquad (b)\ f(x) = \sin^2 x.$$

Solution: (a) Using the Binomial Series,

$$\begin{aligned} f(x) &= \frac{x^2}{\sqrt{2}}\left(1+\frac{x}{2}\right)^{-1/2} = \frac{x^2}{\sqrt{2}}\sum_{n=0}^{\infty}\binom{-1/2}{n}\frac{x^n}{2^n} \\ &= \frac{x^2}{\sqrt{2}}\sum_{n=0}^{\infty}(-1)^n\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n n!}\frac{x^n}{2^n} \\ &= \sum_{n=0}^{\infty}(-1)^n\frac{1\cdot 3\cdot 5\cdots(2n-1)}{4^n\sqrt{2}\,n!}x^{n+2}. \end{aligned}$$

(b) Since $\sin^2 x = (1-\cos 2x)/2$, using the Taylor series for $\cos 2x$ we find that

$$f(x) = \frac{1}{2} - \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(2x)^n = \frac{1}{2} + \sum_{n=0}^{\infty}\frac{(-1)^{n-1}2^{n-1}}{(2n)!}x^n.$$

□

Problem 7. *Solve the following differential equations:*

$$(a)\ \frac{dy}{dx} = e^{2x}y - y + e^{2x} - 1 \qquad (b)\ t^2\frac{dy}{dt} + 3ty = \sqrt{1+t^2},\ t>0.$$

Solution: (a) After factorizing the right-hand side of the differential equation, we immediately realize that it is a separable equation:

$$\frac{dy}{dx} = (e^{2x}-1)(y+1).$$

Therefore

$$\ln|y+1| + C = \int \frac{dy}{y+1} = \int (e^{2x}-1)\,dx = \frac{1}{2}e^{2x} - x.$$

Hence $|y+1| = A\exp(e^{2x}/2-x)$, where $A>0$, which is equivalent to $y = A\exp(e^{2x}/2-x)-1$ where $A \neq 0$. Since the constant function $y=-1$ is also a solution of the differential equation given, we conclude that the set of solutions can be described as follows

$$y = Ae^{e^{2x}/2-x} - 1, \text{ where } A \in \mathbb{R}.$$

(b) Notice that this is a linear differential equation whose integrating factor is

$$I(t) = e^{\int 3/t\,dt} = e^{3\ln|t|} = t^3.$$

Hence $(t^3 y) = t\sqrt{1+t^2}$, which implies that

$$t^3 y = \int t\sqrt{1+t^2}\,dt = \frac{1}{2}\int \sqrt{u}\,du = \frac{1}{2}\int \sqrt{t}\,dt = \frac{1}{3}t^{3/2} + C.$$

Thus, the set of solutions is given by

$$y = \frac{1}{3t^3}(t^{3/2} + C), \text{ where } C \in \mathbb{R}.$$

□

Problem 8. *Suppose a population growth according to a logistic model with initial population* 1000 *and carrying capacity* 10,000*. If the population grows to* 2500 *after one year, what will the population be after another three years?*

Solution: The logistic model is described by the differential equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right),$$

where k is a constant and M is the carrying capacity. In the current case, $M = 10^4$. After solving the integrals in equation resulting from separating variables, namely

$$\int \frac{dP}{P(1 - \frac{P}{10^4})} = \int k dt,$$

we find that

$$P(t) = \frac{10^4}{1 + Ae^{-kt}}, \text{ where } A = \frac{M - P(0)}{P(0)}.$$

Since $P(0) = 1000$, it follows that $A = 9$ and

$$P(t) = \frac{10^4}{1 + 9e^{-kt}}. \tag{0.2}$$

We can use the fact that $P(1) = 2500$ to find k since from equation (0.2) we get

$$k = -\frac{1}{t} \ln \left(\frac{1}{9} \left(\frac{10^4}{P(t)} - 1 \right) \right).$$

Therefore $k = \ln 3$ and so the equation modeling the population under study is

$$P(t) = \frac{10^4}{1 + 27e^{-t}},$$

The population after another three years will be $P(4)$. □

Problem 9. *Find the general solution of the following differential equation:*

$$y'' - 4y' + 5y = 6e^{-x}.$$

Solution: The auxiliary equation, $z^2 - 4z + 5 = 0$ has to conjugate complex roots, namely $2 - i$ and $2 + i$, the set of solutions of the associated homogeneous equation, $y'' - 4y' + 5y = 0$, is

$$ae^{2x} \cos x + be^{2x} \sin x \quad \text{where } a, b \in \mathbb{R}.$$

To find a particular solution of the non-homogeneous, substitute $y_0 = Ae^{-x}$ in the initial equation, and solve it to obtain $A = 3/5$. Hence the set of solutions for the given differential equation is

$$\frac{3}{5}e^{-x} + ae^{2x}\cos x + be^{2x}\sin x \quad \text{where } a, b \in \mathbb{R}.$$

□

Problem 10. *Use power series to solve the following initial-value problem:*

$$y'' + xy = 0, \quad y(0) = 1 \quad y'(0) = 0. \tag{0.3}$$

Solution: Substituting

$$y = \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2}$$

in equation (0.3) we obtain

$$\begin{aligned} 0 &= \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + x\sum_{n=0}^{\infty} a_n x^n \\ &= 2a_2 + \sum_{n=0}^{\infty} a_{n+3}(n+3)(n+2)x^{n+1} + \sum_{n=0}^{\infty} a_n x^{n+1} \\ &= 2a_2 + \sum_{n=0}^{\infty} \big(a_{n+3}(n+3)(n+2) + a_n\big)x^{n+1}. \end{aligned}$$

Therefore $a_2 = 0$ and for every $n \geq 0$ we can compute every coefficient a_{n+3} in terms of a_0 and a_1 by using the formula

$$a_{n+3} = \frac{-1}{(n+3)(n+2)}a_n.$$

Therefore the set of solutions is

$$y = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(3n)_1!}x^{3n} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(3n+1)_2!}x^{3n+1},$$

where $(3n)_1!$ is the product of all integers from 1 to $3n$ omitting those that are of the form $3k+1$, and $(3n+1)_2!$ is the product of all integers from 1 to $3n+1$ omitting those that are of the form $3k+2$. □