PROBLEM SET 9: MEAN VALUE THEOREM AND L'HOPITAL'S RULE

Note: Most of the problems were taken from the textbook [1].

Problem 1. *Show that the following equations have exactly one root:*

$$2x + \cos x = 0 \qquad \textit{and} \qquad x^3 + e^x = 0.$$

Problem 2. *Show that a polynomial of degree* 3 *has at most three real roots.*

Problem 3. *Does there exist a function* f *such that* $f(0) = -1$*,* $f(2) = 4$*, and* $f'(x) \leq 2$ *for all* x*?*

Problem 4. *Show that* $\sin x < x$ *if* $0 < x < 2\pi$*.*

Problem 5. *Argue that* $|\sin a - \sin b| \leq |a - b|$ *for all* $a, b \in \mathbb{R}$*.*

Problem 6. *Prove that if* f *is a differentiable function such that* $f'(x) \neq 1$ *for all real numbers* x*, then there exists at most one* $x \in \mathbb{R}$ *such that* $f(x) = x$*.*

Problem 7. *Find the following limits.*

a) $\lim_{x\to\infty} \frac{e^{x/10}}{x^3}$;

b) $\lim_{x\to\infty} \frac{(\ln x)^2}{x}$;

c) $\lim_{x\to 0} \frac{x+\sin x}{x+\cos x}$;

d) $\lim_{x\to\infty} x\tan(1/x)$;

e) $\lim_{x\to\infty} x^{1/x}$;

f) $\lim_{x\to 0}(\csc x - \cot x)$;

g) $\lim_{x\to\infty} \left(\frac{2x-3}{2x+5}\right)^{2x+1}$.

Problem 8. *Evaluate*

$$\lim_{x\to\infty} \left[x - x^2 \ln\left(\frac{1+x}{x}\right)\right].$$

Problem 9. *Suppose that* f *is a positive function such that* $\lim_{x\to a} f(x) = 0$ *and* $\lim_{x\to a} g(x) = \infty$*. Show that* $\lim_{x\to a} f(x)^{g(x)} = 0$*.*

References

[1] J. Stewart: *Single Variable Calculus* 8th Edition, Cengage Learning, Boston 2015.