IDEAL THEORY IN PRÜFER DOMAINS

FELIX GOTTI

Krull's Theorems and Artin-Rees Lemma

The purpose of this lecture is to prove three important results for Noetherian rings (two of them due to Wolfgang Krull); these are Krull's Intersection Theorem, Artin-Rees Lemma, and Krull's Principal Ideal Theorem.

Krull's Intersection Theorem. The following proposition is a version of Krull Intersection Theorem for Noetherian rings. The proof that we discuss does not use primary decomposition, and was given by H. Perdry in [2].

Proposition 1. *Let R be a commutative ring with identity that is Noetherian, and let I be a ideal of R. Then there exists $r \in I$ such that $(1-r)\bigcap_{n \in \mathbb{N}} I^n = (0)$.*

Proof. Write $I = (a_1, \dots, a_\ell)$ and $\bigcap_{n \in \mathbb{N}} I^n = (b_1, \dots, b_k)$. Now fix $j \in [\![1,k]\!]$. For every $n \in \mathbb{N}$, the fact that $b_j \in I^n$ guarantees the existence of a homogeneous polynomial $p_n \in R[x_1, \dots, x_\ell]$ of degree n such that $b_j = p_n(a_1, \dots, a_\ell)$. For each $n \in \mathbb{N}$, consider the ideal $J_n = (p_1, \dots, p_n)$ of $R[x_1, \dots, x_\ell]$. Since the chain of ideals $(J_n)_{n \in \mathbb{N}}$ is ascending and $R[x_1, \dots, x_\ell]$ is a Noetherian ring by Hilbert Basis Theorem, there is an $n \in \mathbb{N}$ such that $J_{n+1} = J_n$. In particular, p_{n+1} belongs to J_n. As a result, we can take polynomials $q_1, \dots, q_n \in R[x_1, \dots, x_\ell]$ such that $p_{n+1} = \sum_{i=1}^n q_i p_{n+1-i}$. Observe that there is no loss of generality in assuming that q_d is a homogeneous polynomial of degree d for every $d \in [\![1,n]\!]$, and we do so. After evaluating both sides of $p_{n+1} = \sum_{i=1}^n q_i p_{n+1-i}$ at $(x_1, \dots, x_\ell) = (a_1, \dots, a_\ell)$, we see that

$$b_j = (q_1(a_1, \dots, a_\ell) + \dots + q_{n+1}(a_1, \dots, a_\ell))b_j = r_j b_j$$

for some $r_j \in I$ (here we have used the fact that $q_1, \dots, q_{n+1}$ are homogeneous polynomials of positive degree). Therefore, for every $j \in [\![1,k]\!]$, we have found $r_j \in I$ satisfying that $(1-r_j)b_j = 0$. Then the product $(1-r_1)\cdots(1-r_k)$ annihilates b_j for every $j \in [\![1,k]\!]$. Hence $(1-r)\bigcap_{n \in \mathbb{N}} I^n = (0)$ when $r = 1-(1-r_1)\cdots(1-r_k)$. □

The previous proposition is specially useful in the context of integral domains and local rings.

Theorem 2 (Krull's Intersection Theorem)**.** *Let R be a Noetherian domain or a Noetherian local ring, and let I be a proper ideal of R. Then $\bigcap_{n \in \mathbb{N}} I^n = (0)$.*

Proof. When R is an integral domain, the statement of the theorem follows immediately from Proposition 1. On the other hand, suppose that R is a local ring with maximal ideal M, and set $J = \bigcap_{n \in \mathbb{N}} M^n$. Since R is Noetherian, J is a finitely generated R-module. As $MJ = J$, it follows from Nakayama's Lemma that $J = (0)$. Hence $\bigcap_{n \in \mathbb{N}} I^n \subseteq \bigcap_{n \in \mathbb{N}} M^n = (0)$. □

The conclusion of Krull's Intersection Theorem does not hold, in general, for Noetherian rings, as the following example indicates.

Example 3. Consider the ring $R = \mathbb{Z}/6\mathbb{Z}$. Since R is finite, it is Noetherian. On the other hand, R is not local (both $(2+6\mathbb{Z})$ and $(3+6\mathbb{Z})$ are maximal ideals of R) and R is not an integral domain ($2+6\mathbb{Z}$ and $3+6\mathbb{Z}$ are both nonzero zero-divisors). Finally, we observe that $I = (2+6\mathbb{Z})$ is an idempotent ideal and, therefore, $2+6\mathbb{Z} \in \bigcap_{n \in \mathbb{N}} I^n$.

Artin-Rees Lemma. We proceed to prove the Artin-Rees Lemma, which also deals with ideals in Noetherian rings.

Theorem 4 (Artin-Rees Lemma)**.** *Let R be a Noetherian ring, and let I, J, and K be ideals of R. Then there exist $m \in \mathbb{N}$ such that*

$$I^n J \cap K = I^{n-m}(I^m J \cap K) \qquad (0.1)$$

for every $n \in \mathbb{N}$ with $n \ge m$.

Proof. Write $I = (a_1, \dots, a_k)$. For each $n \in \mathbb{N}_0$, let H_n be the set consisting of homogeneous polynomials $f \in R[x_1, \dots, x_n]$ of degree n with $f(a_1, \dots, a_k) \in I^n J \cap K$. Now let I' be the homogeneous ideal generated by the set $H := \bigcup_{n \in \mathbb{N}_0} H_n$. In light of Hilbert Basis Theorem, we can write $I' = (f_1, \dots, f_t)$ for some $f_1, \dots, f_t \in R[x_1, \dots, x_k]$. Since I' is a homogeneous ideal, we can assume that $f_1, \dots, f_t$ are homogeneous polynomials. For each $i \in [\![1, t]\!]$, set $d_i := \deg f_i$, and then set $m = \max\{d_i : i \in [\![1, t]\!]\}$ and fix $n \in \mathbb{N}$ with $n \ge m$.

To argue the inclusion $I^n J \cap K = I^{n-m}(I^m J \cap K)$, take $a \in I^n J \cap K$. As $a \in I^n$, we can pick a polynomial $f \in H_n$ such that $a = f(a_1, \dots, a_k)$. Now write $f = \sum_{i=1}^t g_i f_i$ for some $g_1, \dots, g_t \in R[x_1, \dots, x_t]$. Since f is homogeneous of degree n, there is no loss of generality in assuming that g_i is homogeneous of degree $n - d_i$ for every $i \in [\![1, t]\!]$. Then the fact that

$$a = f(a_1, \dots, a_k) = \sum_{i=1}^t g_i(a_1, \dots, a_k) f_i(a_1, \dots, a_k) \in \sum_{i=1}^t I^{n-d_i}(I^{d_i} J \cap K),$$

along with

$$\sum_{i=1}^t I^{n-d_i}(I^{d_i} J \cap K) \subseteq I^{n-m} \sum_{i=1}^t (I^m J) \cap I^{m-d_i} K) \subseteq I^{n-m}(I^m J \cap K),$$

allows us to conclude that $a \in I^{n-m}(I^m J \cap K)$. Hence the direct inclusion of (0.1) holds. The reverse inclusion follows easily: $I^{n-m}(I^m J \cap K) \subseteq I^n J \cap I^{n-m} K \subseteq I^n J \cap K$. Hence (0.1) holds for every $n \ge m$. □

Krull's Principal Ideal Theorem. Our next goal is to prove Krull's Principal Ideal Theorem (Krull's Hauptidealsatz), which states that, in a Noetherian ring, every minimal prime ideal over a principal ideal has height at most one.

Let R be a commutative ring with identity. The *height* of a prime ideal P of R, which is denoted by $\text{ht}(P)$, is the maximum $h \in \mathbb{N}_0 \cup \{\infty\}$ such that there is a chain

$$P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_h = P,$$

where $P_0, \dots, P_h$ are prime ideals of R. Given an ideal I of R, recall that a minimal prime ideal over I is a prime ideal P containing I such that for every prime ideal Q with $I \subseteq Q \subseteq P$ the equality $Q = P$ holds. Finally, we need the following lemma.

Lemma 5. *Let R be a Noetherian ring, and let P be a prime ideal of R. For every $n \in \mathbb{N}$, set $P^{(n)} := P^n R_P \cap R$. Then $P^{(n)} R_P = P^n R_P$.*

Proof. Exercise. □

The ideal $P^{(n)}$ in the previous lemma is called the *n-th symbolic power* of P. We are now in a position to prove Krull's Principal Ideal Theorem.

Theorem 6 (Krull's Principal Ideal Theorem)**.** *Let R be a Noetherian domain, and let I be a proper principal ideal of R. Then each minimal prime ideal over I has height at most one.*

Proof. Let P be a minimal prime ideal over I. After localizing R at P if necessary, all the relevant data is preserved and we can further assume that R is a local ring with maximal ideal P. Suppose, by way of contradiction, that $\text{ht}(P) \ge 2$. Let Q_0 and Q be prime ideals in R such that $Q_0 \subsetneq Q \subsetneq P$. Observe that if we replace R by R/Q_0, then we can assume that R is a Noetherian domain that is local with maximal ideal P satisfying that P is a minimal prime over I and $(0) \subsetneq Q \subsetneq P$.

Take $a \in R$ such that $I = Ra$ and, for each $n \in \mathbb{N}$, set $Q^{(n)} = Q^n R_Q \cap R$. Observe that $Q^{(n)}$ is a Q-primary ideal for every $n \in \mathbb{N}$. The quotient ring R/Ra has only one prime ideal, namely, P/Ra. Therefore it is a zero-dimensional Noetherian (local) ring, and so it is also an Artinian ring. As a result, the chain of ideals $((Q^{(n)} + Ra)/Ra)_{n \in \mathbb{N}}$ of R/Ra eventually stabilizes, and so there is an $N \in \mathbb{N}$ such that $Q^{(n)} + Ra = Q^{(n+1)} + Ra$ for every $n \ge N$.

Fix $n \ge N$, and then take $q_n \in Q^{(n)}$. Since $Q^{(n)} \subseteq Q^{(n+1)} + Ra$, we can write $q_n = q_{n+1} + ra$ for some $q_{n+1} \in Q^{(n+1)}$ and $r \in R$. Note that $ra = q_n - q_{n+1} \in Q^{(n)}$. In addition, $a \notin Q$ because P is a minimal prime over Ra in R. This, along with the fact that $Q^{(n)}$ is Q-primary, ensures that $r \in Q^{(n)}$. As a consequence, $Q^{(n)} \subseteq Q^{(n+1)} + Q^{(n)} a$, which implies that $Q^{(n)} = Q^{(n+1)} + Q^{(n)}a$. Therefore the R-module $M = Q^{(n)}/Q^{(n+1)}$ satisfies that $M = aM$. So it follows from Nakayama's Lemma that $M = \{0\}$, whence $Q^{(n)}/Q^{(n+1)} = \{0\}$.

Thus, for each $n \ge N$ the equality $Q^{(n)} = Q^{(N)}$ holds, and so $Q^n R_Q = Q^N R_Q$ by virtue of Lemma 5. Take a nonzero $q \in Q$. As R is an integral domain, q^N is a nonzero element of $Q^n R_Q$ for every $n \in \mathbb{N}$. Now since R_Q is a Noetherian local ring with maximal ideal QR_Q, the fact that $q^N \in \bigcap_{n\in\mathbb{N}} Q^n R_Q$ generates a contradiction with Krull's Intersection Theorem, which completes the proof. □

The following related statement follows as a consequence of Krull's Principal Ideal Theorem.

Corollary 7. *Let R be a Noetherian ring, and suppose that $a \in R$ is not a zero-divisor. Prove that $\text{ht}(P) = 1$ for every minimal prime ideal over Ra.*

Proof. Exercise. □

Exercises

Exercises 1 (Nagata's Idealization Trick)**.** *Let R be any commutative ring identity, and let M be a module over R. For the abelian group $S := R \times M$, prove the following statements.*

(1) *S is a commutative ring with identity under the multiplication operation*
$$(r_1, m_1)(r_2, m_2) := (r_1 r_2, r_1 m_2 + r_2 m_1).$$
(2) *$I := \{0\} \times M$ is an ideal of S satisfying that $S/I \cong R$ and $I^2 = (0)$.*
(3) *Every prime ideal of S has the form $P \times M$ for some prime ideal P of R.*
(4) *S is a local ring provided that R is a local ring.*
(5) *S is Noetherian provided that both R and M are Noetherian.*

Exercises 2 (Krull's Intersection Theorem for Modules)**.** *Let R be a Noetherian local ring with maximal ideal P, and let M be a finitely generated module over R. Prove that $\bigcap_{n\in\mathbb{N}} P^n M = 0$. [Hint: Use Nagata's Idealization Trick.]*

Exercises 3. *Let R be a Noetherian ring, and let P be a prime ideal of R. Prove that $P^{(n)} R_P = P^n R_P$ for every $n \in \mathbb{N}$.*

Exercises 4. *Let R be a Noetherian ring, and suppose that $a \in R$ is not a zero-divisor. Prove that $\text{ht}(P) = 1$ for every minimal prime ideal over Ra. [Hint: Argue that in a Noetherian ring every minimal prime ideal consists of zero-divisors.]*

References

[1] W. Krull: *Primidealketten in allgemeinen Ringbereichen*, Berlin-Leipzig, 1928.
[2] H. Perdry: *An elementary proof of Krull's Intersection Theorem*, Amer. Math. Monthly **111** (2004) 356–357.

Department of Mathematics, MIT, Cambridge, MA 02139
Email address: fgotti@mit.edu