IDEAL THEORY AND PRÜFER DOMAINS

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Primary Decompositions in Noetherian Rings

The intersection of primary ideals in not necessarily primary; indeed, even in $\mathbb{Z}$ the non-primary ideal $6\mathbb{Z}$ is the intersection of the prime ideals $2\mathbb{Z}$ and $3\mathbb{Z}$. We have seen that primary ideals in $\mathbb{Z}$ are precisely those generated by powers of primes and, therefore, it follows from the Fundamental Theorem of Arithmetic that each proper ideal of $\mathbb{Z}$ is the intersection of finitely many primary ideals. The main purpose of this lecture is to prove the Lasker-Noether Decomposition Theorem, which states that every proper ideal in a Noetherian ring can be expressed as an *irredundant* intersection of finitely many primary ideals and that any two of such decompositions yield the same set of radical ideals. This result was first proved by E. Lasker in the context of polynomial rings, and the proof was then simplified and generalized by E. Noether. Throughout this lecture, R is a commutative ring with identity.

Let P be a prime ideal of R. Recall that an ideal Q of R is called P-primary if Q is primary and $\operatorname{Rad} Q = P$. Although the intersection of primary ideal is not necessarily primary, it turns out that the intersection of P-primary ideals is a P-primary ideal.

Proposition 1. *If $Q_1, \dots, Q_n$ are P-primary ideals of R for some prime ideal P, then $\bigcap_{j=1}^n Q_j$ is also a P-primary ideal.*

Proof. Set $Q := \bigcap_{j=1}^n Q_j$, and let us verify that $\operatorname{Rad} Q = P$. Since $Q_i \subseteq P$ for every $i \in [\![1, n]\!]$, it follows that $Q \subseteq P$, which implies that $\operatorname{Rad} Q \subseteq \operatorname{Rad} P = P$. To argue the reverse inclusion, take a prime ideal P' containing Q. Since $Q_1 \cdots Q_n \subseteq Q \subseteq P'$, it follows that $Q_i \subseteq P'$ for some $i \in [\![1, n]\!]$. Thus, $P = \operatorname{Rad} Q_i \subseteq P'$. As a consequence, $P \subseteq \operatorname{Rad} Q$. Hence $\operatorname{Rad} Q = P$.

We proceed to prove that Q is a primary ideal. To do so, take $r, s \in R$ such that $rs \in Q$ and $r \notin Q$. Therefore $r \notin Q_j$ for some $j \in [\![1, n]\!]$. Since $rs \in Q_j$ and Q_j is a primary ideal, there exists $n \in \mathbb{N}$ such that $s^n \in Q_j$. This implies that $s \in \operatorname{Rad} Q_j = P = \operatorname{Rad} Q$, and so $s^m \in Q$ for some $m \in \mathbb{N}$. Hence the ideal Q is primary. $\square$

A proper ideal I of R is *irreducible* if for any ideals J and K in R such that $I = J \cap K$ either $J = I$ or $K = I$. One can readily verify that every prime ideal is irreducible. The converse does not hold even for PIDs.

Example 2. Consider the ideal $4\mathbb{Z}$ of $\mathbb{Z}$ and write $4\mathbb{Z} = a\mathbb{Z} \cap b\mathbb{Z}$ for some ideals $a\mathbb{Z}$ and $b\mathbb{Z}$. After assuming that a and b are positive, we see that $a, b \in \{1, 2, 4\}$ and $\max\{a, b\} = 4$. Thus, $a\mathbb{Z} = 4\mathbb{Z}$ or $b\mathbb{Z} = 4\mathbb{Z}$. We conclude that the ideal $4\mathbb{Z}$ is irreducible even though it is not prime.

It turns out that, in the context of Noetherian rings, every irreducible ideal is primary.

Lemma 3. *In a Noetherian ring, every irreducible ideal is primary.*

Proof. Let R be Noetherian ring, and let Q be an irreducible ideal of R. Take $a, b \in R$ such that $ab \in Q$ but $b \notin Q$. For each $n \in \mathbb{N}$, consider the colon ideal

$$A_n := (Q : (a^n)) = \{r \in R : ra^n \in Q\}.$$

One can readily see that $(A_n)_{n \in \mathbb{N}}$ is an ascending chain of ideals. Since R is Noetherian, there is an $n \in \mathbb{N}$ such that $A_m = A_n$ for every $m \geq n$. Now consider the ideals $I := (a^n) + Q$ and $J := (b) + Q$. It is clear that $Q \subseteq I \cap J$. To argue the reverse inclusion, take $y \in I \cap J$ and write $y = ra^n + q$ for some $r \in R$ and $q \in Q$. As $aJ \subseteq Q$, it follows that $ay \in Q$. Therefore $ra^{n+1} = ay - aq \in Q$. This implies that $r \in A_{n+1} = A_n$, and so $y = ra^n + q \in Q$. Thus, $Q = I \cap J$. Because Q is irreducible, $Q = I$ or $Q = J$. Now the fact that $b \notin Q$ ensures that $Q = I$, and so $a^n \in Q$. Hence Q is a primary ideal. $\square$

Even in the context of Noetherian rings not every primary ideal is irreducible, as the following example shows.

Example 4. Consider the ideal $Q = (x, y)^2$ of $\mathbb{Q}[x, y]$. Since Q is a power of the maximal ideal (x, y), it must be primary. However, Q is not irreducible because it is the intersection of the ideals $(x, y^2) = (x) + (x, y)^2$ and $(y, x^2) = (y) + (x, y)^2$ (Exercise 2).

Remark 5. Even in the context of Noetherian rings, the notions of irreducible and radical ideals are not comparable. Observe that the radical ideal $6\mathbb{Z}$ of $\mathbb{Z}$ is not irreducible because it can be written as $6\mathbb{Z} = 2\mathbb{Z} \cap 3\mathbb{Z}$. On the other hand, we have seen in Example 2 that the non-radical ideal $4\mathbb{Z}$ of $\mathbb{Z}$ is irreducible.

An ideal I of R has a *primary decomposition* if $I = \bigcap_{j=1}^n Q_j$ for some primary ideals $Q_1, \dots, Q_n$, in which case, $\{\operatorname{Rad} Q_i : 1 \leq j \leq n\}$ is called the *set of radical ideals* of the decomposition. Such a decomposition is called *irredundant* if the radicals of the ideals $Q_1, \dots, Q_n$ are all distinct and $\bigcap_{j \neq k} Q_j \not\subseteq Q_k$ for any $k \in [\![1, n]\!]$. If every ideal in R has an irredundant primary decomposition, then R is called a *Lasker ring*. An *associated prime ideal* of I is a prime ideal of the form $(I : Rc)$ for some $c \in R \setminus I$. It turns out that when R is a Noetherian ring, every proper ideal I has an irredundant primary decomposition and also that the set of radical ideals of such a decomposition coincides with the set of associated prime ideals of I. In particular, every Noetherian ring is a Lasker ring. This important result is due to E. Lasker and E. Noether.

Theorem 6. *Every proper ideal I in a Noetherian ring R has an irredundant primary decomposition. In addition, the set of radical ideals of any irredundant primary decomposition of I coincides with its set of associated prime ideals.*

Proof. Let R be a Noetherian ring. By virtue of Lemma 3, in order to prove that every proper ideal of R has a primary decomposition, it suffices to argue that every proper ideal of R is the finite intersection of irreducible ideals. Suppose, by way of contradiction, that this is not the case, and let $\mathscr{S}$ be the set of all the ideals of R that cannot be written as finite intersections of irreducible ideals. Since R is a Noetherian ring and the set $\mathscr{S}$ of ideals of R is nonempty, $\mathscr{S}$ must contain a maximal element J. Since J belongs to $\mathscr{S}$, it is not irreducible and so there exist ideals I_1 and I_2 both properly containing J such that $J = I_1 \cap I_2$. The maximality of J now implies that both I_1 and I_2 can be written as finite intersections of irreducible ideals in R. However, this immediately implies that J can be also written as a finite intersection of irreducible ideals in R, contradicting that J belongs to $\mathscr{S}$. Thus, every proper ideal of R has a primary decomposition.

In addition, one can easily see that any primary decomposition can be turned into an irredundant primary decomposition by dropping unnecessary primary ideals from the intersection and successively replacing all primary ideals with the same radical by their intersection.

Finally, let us show that for every ideal I of R, the set of radical ideals of I is $\{(I : Rc) \in \operatorname{Spec}(R) : c \in R \setminus I\}$. To do so, fix a proper ideal I of R, let $I = \bigcap_{i=1}^n Q_i$ be an irredundant primary decomposition of I, and set $P_j := \operatorname{Rad} Q_j$ for a fixed $j \in [\![1, n]\!]$. Now write $J = \bigcap_{i \neq j} Q_i$, and observe that $I = J \cap Q_j$ is strictly contained in J. Since R is Noetherian, there exists $n \in \mathbb{N}$ such that $P_j^n \subseteq Q_j$, and so $JP_j^n \subseteq J \cap Q_j = I$. Assume that n is the minimum positive integer such that $JP_j^n \subseteq I$. Take $c \in JP_j^{n-1}$ such that $c \notin I$. The fact that $c \in J$, along with $c \notin I$, ensures that $c \notin Q_j$. So if $r \in R$ satisfies $cr \in I \subseteq Q_j$, then the fact that Q_j is primary guarantees that $r \in \operatorname{Rad} Q_j = P_j$. Hence $(I : Rc) \subseteq P_j$. Conversely, observe that $cP_j \subseteq JP_j^n \subseteq I$, which implies that $P_j \subseteq (I : Rc)$. Hence $P_j = (I : Rc)$, as desired.

Now fix $c \in R \setminus I$ with $(I : Rc)$ prime and then set $P := (I : Rc)$. Note that there is a $j \in [\![1, n]\!]$ such that $c \notin Q_j$. Consider the ideal $K := \Pi_{c \notin Q_i} Q_i$. Clearly, $cK \in \bigcap_{i=1}^n Q_i = I$. Therefore $K \subseteq (I : Rc) = P$, and the fact that P is prime ensures that $Q_i \subseteq P$ for some $i \in [\![1, n]\!]$ with $c \notin Q_i$. Thus, $\operatorname{Rad} Q_i \subseteq P$. On the other hand, take $x \in P$, and observe that $cx \in I \subseteq Q_i$. Because Q_i is primary and $c \notin Q_i$, it follows that $x \in \operatorname{Rad} Q_i$. Hence $P = \operatorname{Rad} Q_i$. $\square$

Let us proceed to discuss how primary decompositions behave under localization.

Proposition 7. *Let R be a Noetherian ring, and let $I = \bigcap_{i=1}^n Q_i$ be an irredundant primary decomposition of a proper ideal I of R. Let S be a multiplicative subset of R and assume that there is an $m \in [\![1,n]\!]$ such that $S \cap \operatorname{Rad} Q_i = \emptyset$ if and only if $i \in [\![1,m]\!]$. Then*

$$S^{-1}I = \bigcap_{i=1}^m S^{-1}Q_i \quad \text{and} \quad {}^c(S^{-1}I) = \bigcap_{i=1}^m Q_i$$

are irredundant primary decompositions of the ideals $S^{-1}I$ and ${}^c(S^{-1}I)$, respectively, where ${}^c(S^{-1}I)$ is the contraction of $S^{-1}I$ back to R.

Proof. As localization commutes with intersection of ideals, $S^{-1}I = \bigcap_{i=1}^n S^{-1}Q_i$, which implies that $S^{-1}I = \bigcap_{i=1}^m S^{-1}Q_i$ because $S^{-1}Q_i = S^{-1}R$ when $i > m$. Now note that if for some $j \in [\![1,m]\!]$ the inclusion $\bigcap_{i \neq j} S^{-1}Q_i = S^{-1}\big(\bigcap_{i \neq j} Q_i\big) \subseteq S^{-1}Q_j$ held, then $\bigcap_{i \neq j} Q_i \subseteq {}^c(S^{-1}(\bigcap_{i \neq j} Q_i)) \subseteq {}^c(S^{-1}Q_j) = Q_j$. Hence $S^{-1}I = \bigcap_{i=1}^m S^{-1}Q_i$ is an irredundant primary decomposition. Since the extension of ideals $I \mapsto S^{-1}I$ induces a bijection from the set of primary ideals of R disjoint from S to the set of primary ideals of $S^{-1}R$ (whose inverse is given by $J \mapsto {}^cJ$), after contracting both sides of the equality $S^{-1}I = \bigcap_{i=1}^m S^{-1}Q_i$, we obtain the primary decomposition ${}^c(S^{-1}I) = \bigcap_{i=1}^m Q_i$ in R of the ideal ${}^c(S^{-1}I)$, which is irredundant because $I = \bigcap_{i=1}^n Q_i$ is irredundant. □

Let R be a Noetherian ring, and let $I = \bigcap_{i=1}^n Q_i$ be a primary decomposition of a proper ideal I of R. For each $i \in [\![1,n]\!]$, the ideal Q_i is called the *primary component* of the associated prime ideal $\operatorname{Rad} Q_i$. If an associated prime ideal P of I is minimal in the set of all associated prime ideals of I, then P is called an *isolated prime ideal* of I. The rest of the associated prime ideals of I are called *embedded prime ideals* of I.

Proposition 8. *Let R be a Noetherian ring, and let I be a proper ideal of R. Then the primary components of isolated prime ideals in an irredundant primary decomposition of I are uniquely determined by I (i.e., they do not depend on the primary decomposition).*

Proof. Let $\{P_i : i \in [\![1,n]\!]\}$ be the set of associated prime ideals of I, and let P be an isolated prime ideal of I. Consider the multiplicative subset $S := R \setminus P$ of R, and observe that $S \cap P_i$ is nonempty for every P_i distinct from P. Let $I = \bigcap_{i=1}^n Q_i$ be an irredundant primary decomposition of I with $\operatorname{Rad} Q_1 = P$. It follows from Proposition 7 that $I_P = \bigcap_{i=1}^n (Q_i)_P = (Q_1)_P$, and so $Q_1 = {}^c(I_P)$ because Q_1 is a primary ideal. □

Exercises

Exercise 1. *The Fundamental Theorem of Arithmetic (FTA) states that every nonzero positive integer greater than 1 uniquely factors (up to permutations) into primes. Deduce the FTA from the Noether-Lasker Theorem (Theorem 6).*

Exercise 2.

(1) *Prove that the ideal* $Q = (x, y)^2$ *of* $\mathbb{Q}[x, y]$ *is a primary ideal that is not irreducible (Hint: show that* Q *is the intersection of the ideals* $(x, y^2) = (x) + (x, y)^2$ *and* $(y, x^2) = (y) + (x, y)^2$*).*

(2) *Find a commutative ring with identity having an irreducible ideal that is not primary.*

Exercise 3. *Let* R *be a commutative ring with identity. Prove the following statements.*

(1) *If* Q *be a* P*-primary ideal of* R*, then* $QR[x]$ *is a* $PR[x]$*-primary ideal of* $R[x]$ *lying over* Q*, that is,* $QR[x] \cap R = Q$*.*

(2) *If* $I_1, \dots, I_n$ *are ideals of* R*, then* $\left(\bigcap_{j=1}^n I_j\right)R[x] = \bigcap_{j=1}^n I_j R[x]$*.*

Exercise 4. *Let* K *be an infinite field, and set* $Q_\alpha = (y - \alpha x, x^2)$ *for any* $\alpha \in K$*.*

(1) *Prove that* Q_α *is a primary ideal of* $K[x, y]$ *for all* $\alpha \in K$*.*

(2) *Show that* $(x^2, xy) = (x) \cap Q_\alpha$ *is an irredundant primary decomposition of* (x^2, xy) *in* $K[x, y]$ *for all* $\alpha \in K$*.*

(3) *Find the associate prime ideals of* (x^2, xy) *in* $K[x, y]$*.*

(4) *Prove that* $Q_\alpha \neq Q_\beta$ *for any* $\alpha, \beta \in K$ *with* $\alpha \neq \beta$*.*

(5) *Deduce that an ideal may have infinitely many irredundant primary decompositions (even in the context of Noetherian rings).*

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