IDEAL THEORY AND PRÜFER DOMAINS

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Integer-Valued Polynomials II

The primary purpose of this lecture is to show that $\text{Int}(\mathbb{Z})$ is a Prüfer domain. After doing so, we prove that $\text{Int}(\mathbb{Z})$ satisfies the Skolem property: for any polynomials $f_1, \dots, f_k \in \text{Int}(\mathbb{Z})$ the fact that $\gcd(f_1(n), \dots, f_k(n)) = 1$ for every $n \in \mathbb{Z}$ implies that $c_1(x)f_1(x) + \dots + c_k(x)f_k(x) = 1$ for some $g_1, \dots, g_k \in \text{Int}(\mathbb{Z})$.

Theorem 1. *The ring* $\text{Int}(\mathbb{Z})$ *is a Prüfer domain.*

Proof. Let I be a nonzero finitely generated ideal of $\text{Int}(\mathbb{Z})$, and write $I = (f_1, \dots, f_k)$ for some $f_1, \dots, f_k \in I$. Take $J = (\text{Int}(\mathbb{Z}) : I)$. Since $IJ \subseteq R$, proving that I is invertible amounts to arguing that IJ is not contained in any maximal ideal of $\text{Int}(R)$, that is, $IJ \not\subseteq M_{p,\alpha} = \{f \in \text{Int}(\mathbb{Z}) : f(\alpha) \in p\mathbb{Z}_p\}$ for any $p \in \mathbb{P}$ and $\alpha \in \mathbb{Z}_p$. To do this, fix $p \in \mathbb{P}$ and $\alpha \in \mathbb{Z}_p$, and then set

$$n := \inf\{v_p(f(\alpha)) : f \in I\}. \tag{0.1}$$

Since $I\mathbb{Q}[x]$ is an ideal of $\mathbb{Q}[x]$ it is principal and, therefore, we can take $g(x) \in \mathbb{Q}[x]$ such that $I\mathbb{Q}[x] = g(x)\mathbb{Q}[x]$. We can actually assume that $g(x) \in \mathbb{Z}[x]$, and we do so. For each $i \in [\![1, k]\!]$, there is a nonzero $d_i \in \mathbb{Z}$ such that $d_i f_i(x) \in g(x)\mathbb{Z}[x]$, and so after setting $d = d_1 \cdots d_k$, we obtain that $dI \subseteq g(x)\text{Int}(\mathbb{Z})$. As a result, $(d/g)I$ is a finitely generated ideal of $\text{Int}(\mathbb{Z})$, which is invertible if and only if I is invertible. Observe, in addition, that $(d/g)I$ contains a nonzero constant polynomial. Thus, after replacing I by $(d/g)I$, we can further assume that I contains a nonzero constant polynomial. This implies that $n \in \mathbb{N}_0$.

For each $i \in [\![1, k]\!]$, the continuity of f_i guarantees the existence of a clopen subset U_i of α such that $v_p(f_i(x)) \ge n$ for all $x \in U_i$. Now consider the clopen subset $U := \bigcap_{i=1}^k U_i$ of $\mathbb{Z}_p$ containing α. For all $f \in I$, we can easily see that $v_p(f(x)) \ge n$ for all $x \in U$ and $v_p(f(x)) \ge 0$ for all $x \in \mathbb{Z}_p \setminus U$. Now we can use Stone-Weierstrass Theorem to produce an $h \in \text{Int}(\mathbb{Z})$ with $v_p(h(x)) = 0$ for all $x \in U$ and $v_p(h(x)) = n$ for all $x \in \mathbb{Z}_p \setminus U$. Set $h_0 := h/p^n$, and note that $h_0 \in J$ because $fh_0 \in \text{Int}(\mathbb{Z})$. By n belongs to $\mathbb{N}_0$, there exists $f_0 \in I$ such that $v_p(f_0(\alpha)) = n$. As a consequence, even though $f_0 h_0 \in IJ$, the equality $v_p(f_0(\alpha)h_0(\alpha)) = 0$ implies that $f_0 h_0 \notin M_{p,\alpha}$. Hence $IJ \not\subseteq M_{p,\alpha}$ for any $p \in \mathbb{P}$ and $\alpha \in \mathbb{Z}_p$, and so I is an invertible ideal of $\text{Int}(\mathbb{Z})$. We can now conclude that $\text{Int}(\mathbb{Z})$ is a Prüfer domain. $\square$

Since we have see before that $\operatorname{Int}(\mathbb{Z})$ has Krull dimension two, the fact that $\operatorname{Int}(\mathbb{Z})$ is a Prüfer domain implies that it is not Noetherian. Let us show explicitly that none of the maximal ideals of $\operatorname{int}(\mathbb{Z})$ are finitely generated.

Proposition 2. *Every nonzero prime ideal of* $\operatorname{Int}(\mathbb{Z})$ *is non-finitely generated.*

Proof. Let us argue first that every finitely generated proper ideal of $\operatorname{Int}(\mathbb{Z})$ is contained in infinitely many maximal ideals. To do this, fix a nonzero finitely generated proper ideal $I = (f_1, \dots, f_k)$, where $f_1, \dots, f_k \in \operatorname{Int}(\mathbb{Z})$ are nonzero. Then there exist $p \in \mathbb{P}$ and $\alpha \in \mathbb{Z}_p$ such that I is contained in the maximal ideal $M_{p,\alpha}$. Then $v_p(f_i(\alpha)) \ge 1$ and, by continuity, there exists a clopen subset of $\mathbb{Z}_p$ containing α such that $v_p(f(\beta)) \ge 1$ for all $f \in I$ and $\beta \in U$. Therefore I is contained in $M_{p,\beta}$. Since $M_{p,\beta} \ne M_{p,\beta'}$ for any $\beta, \beta' \in \mathbb{Z}_p$ with $\beta \ne \beta'$, the ideal I is contained in infinitely many maximal ideal.

As a result, the maximal ideals of $\operatorname{Int}(\mathbb{Z})$ cannot be finitely generated. It suffices to argue now that none of the prime ideals $P_q = \operatorname{Int}(\mathbb{Z}) \cap q(x)\mathbb{Q}[x]$, where $q(x)$ is an irreducible of $\mathbb{Q}[x]$, is finitely generated. We have proved before that if P_q is contained in a a maximal ideal $M_{p,\alpha}$ for some $p \in \mathbb{P}$ and $\alpha \in \mathbb{Z}_p$, then $q(\alpha) = 0$. Since the $M_{p,\alpha}$'s are the only maximal ideals of $\operatorname{Int}(\mathbb{Z})$, the ideal P_q is contained in only finitely many maximal ideals and, therefore, it cannot be finitely generated by the argument established in the previous paragraph. □

The Skolem Property. For every ideal I of $\operatorname{Int}(\mathbb{Z})$ and for every $n \in \mathbb{Z}$, set $I(n) := \{f(n) : f \in I\}$. It is clear that $I(n)$ is an ideal of $\mathbb{Z}$ for every $n \in \mathbb{Z}$. It turns out that if I is a finitely generated ideal of $\operatorname{Int}(\mathbb{Z})$ satisfying that $I(n) = \mathbb{Z}$ for every $n \in \mathbb{Z}$, then $I = \operatorname{Int}(\mathbb{Z})$. This property satisfied by $\operatorname{Int}(\mathbb{Z})$ is referred to as the *Skolen property*, as it was first proved by T. Skolem [3].

Proposition 3. *If I is a finitely generated ideal of* $\operatorname{Int}(\mathbb{Z})$ *satisfying that $I(n) = \mathbb{Z}$ for every $n \in \mathbb{Z}$, then $I = \operatorname{Int}(\mathbb{Z})$*

Proof. Let I be a finitely generated ideal of $\operatorname{Int}(\mathbb{Z})$ such that $I(n) = \mathbb{Z}$ for every $n \in \mathbb{Z}$. Assume, towards a contradiction, that I is a proper ideal. Then I is contained in one of the maximal ideals $M_{p,\alpha}$ for some $p \in \mathbb{P}$ and $\alpha \in \mathbb{Z}_p$. Since I is generated by finitely many polynomials in $M_{p,\alpha}$, which are continuous and have p-valuation at least 1, there exists a clopen subset U of $\mathbb{Z}_p$ containing α such that $f(\beta) \in p\mathbb{Z}_p$ for all $f \in I$ and $\beta \in U$, that is, $I \subseteq M_{p,\beta}$ for all $\beta \in U$. Since $\mathbb{Z}$ is dense in $\mathbb{Z}_p$, we can choose $n \in \mathbb{Z} \cap U$ such that $I \subseteq M_{p,n}$. This means that $I(n) \subseteq p\mathbb{Z}$, which is a contradiction. □

It is worth to emphasize that, unlike $\operatorname{Int}(\mathbb{Z})$, the ring $\mathbb{Z}[x]$ does not satisfy the Skolem property. The following example was given by Skolem in [3].

Example 4. Consider the polynomials $f(x) = 3$ and $g(x) = x^2 + 1$ in $\mathbb{Z}[x]$, and let I be the ideal generated by f and g in $\mathbb{Z}[x]$. Since -1 is not a quadratic residue modulo 3, we see that $\gcd(3, n^2 + 1) = 1$ for every $n \in \mathbb{Z}$. Now suppose, by way of contradiction, that $I = \mathbb{Z}[x]$ and write $3a(x) + (x^2 + 1)b(x) = 1$ for some $a(x), b(x) \in \mathbb{Z}[x]$. After evaluating at the imaginary unit i, we obtain that $1/3 = a(i)$, which means that $1/3$ is a Gaussian integer, a contradiction.

Exercises

Exercise 1. *Without using the description of Spec(Int($\mathbb{Z}$)) we have established in the previous lecture, show that the ideal $\{f \in \text{Int}(\mathbb{Z}) : f(0) \in 2\mathbb{Z}\}$ of $\text{Int}(\mathbb{Z})$ is not finitely generated.*

Exercise 2. *[Strong Skolem Property] Let I and J be two finitely generated ideals of $\text{Int}(\mathbb{Z})$ such that $I(n) = J(n)$ for every $n \in \mathbb{Z}$. Prove that $I = J$.*

References

[1] P. J. Cahen and J. L. Chabert: *Integer-Valued Polynomials*, Amer. Math. Soc. Surveys and Monographs, Vol. 48, Providence, 1997.
[2] P. J. Cahen and J. L. Chabert: *What you should know about integer-valued polynomials*, Amer. Math. Monthly **123** (2016) 311–337.
[3] T. Skolem: *Ein Satz über ganzwertige Polynome*, Norske Videnskabers Selskab-Trondheim, Forhandlinger **9** (1936) 111–113.

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