IDEAL THEORY AND PRÜFER DOMAINS

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Almost Dedekind Domains

Throughout this lecture, R is an integral domain. Recall that $\mathrm{qf}(R)$ denotes the quotient field of R.

Definition 1. An integral domain R that is not a field is an *almost Dedekind domain* if R_M is a Dedekind domain for every maximal ideal M of R.

We have excluded fields from the class of almost Dedekind domains, but it is worth noticing that some authors may prefer to allow fields to be almost Dedekind domains. Informally, an almost Dedekind domain is an integral domain that looks locally like a Dedekind domain. In particular, every almost Dedekind domain has Krull dimension one. Observe that we can rephrase the definition of an almost Dedekind domain as an integral domain whose localization at any maximal/prime ideal is a DVR.

It turns out that Dedekind domains are precisely the Noetherian almost Dedekind domains.

Proposition 2. *An integral domain R is a Dedekind domain if and only if R is Noetherian and almost Dedekind.*

Proof. If R is a Dedekind domain, then R is Noetherian and R_M is a local PID for every maximal ideal M of R. As local PIDs are DVRs, R is almost Dedekind, and so the direct implication follows. The reverse implication is also immediate as DVRs are PIDs and we have seen before that Dedekind domains can be characterized as Noetherian domains whose localizations at maximal ideals are PIDs. □

Every overring of an almost Dedekind domain is an almost Dedekind domain.

Proposition 3. *Every overring of an almost Dedekind domain is an almost Dedekind domain.*

Proof. Let R an almost Dedekind domain, and let S be an overring of R. Let Q be a prime ideal of S. Then $P := R \cap Q$ is a prime ideal of R and, therefore, R_P is a Dedekind domain. Since $R \setminus P = R \setminus Q \subseteq S \setminus Q$, we see that S_Q is an overring of R_P. Now the fact that R_P is a Dedekind domain ensures that S_Q is also a Dedekind domain. As Q was chosen to be an arbitrary prime ideal of S, we conclude that S is an almost Dedekind domain. □

Here is a useful characterization of an almost Dedekind domain in terms of its primary ideals.

Proposition 4. *Let R be an integral domain that is not a field. Then R is an almost Dedekind domain if and only if R is one-dimensional and every primary ideal is a power of a prime ideal.*

Proof. For the direct implication, suppose that R is an almost Dedekind domain. It is clear that R is one-dimensional. Let Q be a primary ideal of R. Since $M := \operatorname{Rad} Q$ is a prime ideal, it must be maximal because R is one-dimensional. Then Q_M is an ideal of R_M and, because R_M is a DVR, it follows that $Q_M = M_M^n$ for some $n \in \mathbb{N}$. Since Q is primary, $Q = Q_M \cap R = M_M^n \cap R = M^n$. Hence every primary ideal of R is a power of a prime ideal.

For the reverse implication, assume that R is a one-dimensional domain where each primary ideal is a power of a prime ideal. Let M be a maximal ideal of R. Since R_M is one-dimensional and local, it follows that the radical of every ideal in R_M is M_M. Since M_M is maximal, every ideal of R_M is M_M-primary, and so it is the extension of an M-primary ideal of R. Since every primary ideal of R is a power of a prime ideal, the maximality of M ensures that the contraction of every ideal of R_M is a power of M. Thus, every ideal of R_M has the form M_M^n for some $n \in \mathbb{N}$. Then the poset of ideals of R_M is totally ordered, which means that R_M is a valuation domain. In addition, it is clear that every collection of ideals of R_M has a maximal element, and so R_M is Noetherian. Since R_M is a Noetherian valuation domain, it is a DVR. Hence R is an almost Dedekind domain. □

We proceed to give two characterizations of almost Dedekind domains inside the class of Prüfer domains.

Theorem 5. *For an integral domain R that is not a field, the following conditions are equivalent.*

(a) *R is an almost Dedekind domain.*
(b) *R is a Prüfer domain and $\bigcap_{n\in\mathbb{N}} I^n = (0)$ for each proper ideal I of R.*
(c) *R is a Prüfer domain of dimension one without idempotent maximal ideals.*

Proof. (a) $\Rightarrow$ (b): Prüfer domains are precisely the integral domains whose localizations at maximal ideals are valuation domains. Hence R is a Prüfer domain. Now suppose that I is a proper ideal of R, and let M be a maximal ideal of R containing I. Note that $I \subseteq I_M \subseteq M_M$. Since R_M is a Noetherian domain, it follows from Krull's Intersection Theorem that $\bigcap_{n\in\mathbb{N}} I^n \subseteq \bigcap_{n\in\mathbb{N}} M_M^n = (0)$.

(b) $\Rightarrow$ (c): Let M be a maximal ideal of R. Note that $M^2 \subsetneq M$ as, otherwise, $\bigcap_{n\in\mathbb{N}} M_M^n = M_n \neq (0)$ in the Noetherian domain R_M, which would contradict Krull's Intersection Theorem. Hence none of the maximal ideals of R is idempotent. Let us argue now that $\dim R_M = 1$. Observe that every power of M is an M-primary ideal of R because R is a Prüfer domain, and so $M_M^n \cap R = M^n$ for every $n \in \mathbb{N}$. Therefore

$$\Big(\bigcap_{n\in\mathbb{N}} M_M^n\Big) \cap R = \bigcap_{n\in\mathbb{N}} \big(M_M^n \cap R\big) = \bigcap_{n\in\mathbb{N}} M^n = (0).$$

This implies that $\bigcap_{n\in\mathbb{N}} M_M^n = (0)$. Since R_M is a valuation domain, there is no proper ideal strictly between $(0) = \bigcap_{n\in\mathbb{N}} M_M^n$ and M_M, which means that the only prime ideal of R_M is M_M. Thus, R_M has Krull dimension one for every maximal ideal M, and so R has dimension one.

(c) $\Rightarrow$ (a): Let Q be a primary ideal of R and set $P := \operatorname{Rad} Q$. Since Q is primary and R has dimension one, P is a maximal ideal. Therefore $P \neq P^2$, and so the fact that R is a Prüfer domain, guarantees that $Q = P^n$ for some $n \in \mathbb{N}$. As every primary ideal is a power of a prime ideal, it follows from Proposition 4 that R is an almost Dedekind domain. □

One can also characterize almost Dedekind domains as those integral domains whose multiplicative monoids of nonzero ideals are cancellative.

Proposition 6. *Let R be an integral domain that is not a field. Then R is an almost Dedekind domain if and only if the multiplicative monoid of nonzero ideals of R is cancellative.*

Proof. For the direct implication, suppose that R is an almost Dedekind domain, and take ideals I, J, and K of R such that $I \neq 0$ and $IJ = IK$. Fix now a maximal ideal M of R. After localizing at M, we obtain $I_M J_M = I_M K_M$. Since R_M is a Dedekind domain and I_M is a nonzero ideal, $J_M = K_M$. Therefore $J_M = K_M$ for every maximal ideal M of R, which implies that $J = K$. Hence the monoid of nonzero ideals of R is cancellative.

Conversely, suppose that the monoid of nonzero ideals of R is cancellative. In particular, every nonzero finitely generated ideal of R is cancellative and, therefore, R is a Prüfer domain. Also, the fact that every nonzero ideal is cancellative immediately implies that no maximal ideal of R is idempotent. We proceed to show that $\dim R = 1$. To do this, take a prime ideal P of R, and then take $x \in R \setminus P$. Note that $(P + (x))^3 = (P + (x))(P + (x^2))$, so the fact that $P + (x)$ is cancellative ensures that $(P + (x))^2 = P^2 + (x^2)$, and so that $xP \subseteq P^2 + (x^2)$. Then for every $y \in P$, we can take $z \in P^2$ and $r \in R$ such that $xy = z + rx^2$. Then $rx^2 = xy - z \in P$, and so the fact that $x^2 \notin P$ implies that $r \in P$. Hence $xP \subseteq P^2 + (x^2)P$, which implies that $x \in P + (x^2)$. Then we can take $s \in R$ such that $x(1 - sx) \in P$ and, as $x \notin P$, it follows that $1 - sx \in P$. This implies that $R = P + (x)$. Thus, P must be maximal, and so $\dim R = 1$. □

Exercises

Exercise 1. *Let R be an integral domain that is not a field. Prove that R is an almost Dedekind domain if and only if every ideal of R whose radical is prime is a power of its radical.*

Exercise 2. *Let R be an almost Dedekind domain. Prove that R is a Dedekind domain if and only if every nonzero proper ideal is contained in only finitely many maximal ideals. Show that if R has only finitely many maximal ideals, then R is a PID.*

Exercise 3. *Let R be an integral domain with quotient field K, and let L be a finite extension of K. Let T be the integral closure of R in L. Prove that if R is an almost Dedekind domain, then T is also an almost Dedekind domain.*

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