# IDEAL THEORY AND PRÜFER DOMAINS

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## Prüfer Domains II

Throughout this lecture, $R$ is an integral domain. Recall that $\operatorname{qf}(R)$ denotes the quotient field of $R$.

**Primary Ideals.** In the first part of this lecture we extend to Prüfer domains some of the results on primary ideals that we have already established for valuation domains.

We have seen that if $R$ is a valuation domain with a prime ideal $P$ and a $P$-primary ideal $Q$, then $Q = (x)Q$ for every $x \in R \setminus P$. If $R$ is a Prüfer domain instead, then $Q = (Q + (x))Q$ under the same conditions.

**Lemma 1.** *Let $R$ be a Prüfer domain, and let $P$ be a prime ideal of $R$. If $Q$ is a $P$-primary ideal, then $Q = (Q + (x))Q$ for every $x \in R \setminus P$.*

*Proof.* Suppose that $Q$ is a $P$-primary ideal of $R$, and take $x \in R \setminus P$. Let $M$ be a maximal ideal. If $Q \nsubseteq M$, then both ideal extensions $Q_M$ and $Q_M^2$ are zero, and so it is clear that $Q_M = ((Q + (x))Q)_M$. On the other hand, suppose that $Q \subseteq M$. In this case, $P = \operatorname{Rad} Q \subseteq M$ and, therefore, $x \notin P_M$. Since $R$ is a Prüfer domain, $R_M$ is a valuation domain, and so the equality $Q_M = (x)_M Q_M$ holds. Therefore

$$Q_M \subseteq Q_M^2 + (x)_M Q_M = ((Q + (x))Q)_M.$$

Thus, in this case we also obtain that $Q_M = ((Q+(x))Q)_M$. Since the maximal ideal $M$ was chosen arbitrarily, we conclude that $Q = (Q + (x))Q$. □

We have also come across the following lemma in the context of valuation domains.

**Lemma 2.** *Let $R$ be a Prüfer domain, and let $Q$ be a primary ideal of $R$. If $I$ is an ideal of $R$ such that $I \subsetneq \operatorname{Rad} Q$, then $I^n \subseteq Q$ for some $n \in \mathbb{N}$.*

*Proof.* Take an ideal $I$ of $R$ such that $I$ is strictly contained in $P := \operatorname{Rad} Q$. Since $Q$ is primary, $P$ is prime, and so $R_P$ is a valuation domain. Suppose, by way of contradiction, that $I^n \nsubseteq Q$ for any $n \in \mathbb{N}$. From this we can infer that $I_P^n \nsubseteq Q_P$ for any $n \in \mathbb{N}$. Now the fact that $R_P$ is a valuation domain ensures that $Q_P \subseteq \bigcap_{n \in \mathbb{N}} I_P^n$, which is prime. Hence $P_P = \operatorname{Rad} Q_P \subseteq \bigcap_{n \in \mathbb{N}} I_P^n \subseteq I_P$, and so $\operatorname{Rad} Q \subseteq I$, which contradicts that $I \subsetneq \operatorname{Rad} Q$. □

Further fundamental results about primary ideals of Prüfer domain are encapsulated in the following theorem.

**Theorem 3.** *Let $R$ be a Prüfer domain, and let $P$ be a prime ideal of $R$. Then the following statements hold.*

1. *The product of $P$-primary ideals is a $P$-primary ideal.*
2. *If $Q$ is a $P$-primary ideal, then $\bigcap_{n \in \mathbb{N}} Q^n$ is prime.*
3. *If $P$ is not idempotent, then the $P$-primary ideals of $R$ are the powers of $P$.*
4. *The intersection of all $P$-primary ideals of $R$ is a prime ideal that contains each prime ideal properly contained in $P$.*

*Proof.* (1) Since we have already proved this statement in the context of valuation domains, we only need to use the one-to-one correspondence between $P$-primary ideals of $R$ and $P_P$-primary ideals of $R_P$ and the fact that $R_P$ is a valuation domain.

(2) Let $Q$ be a $P$-primary ideal of $R$, and take $x, y \in R$ with $xy \in I := \bigcap_{n \in \mathbb{N}} Q^n$. Since $R_P$ is a valuation domain, $\bigcap_{n \in \mathbb{N}} Q_P^n$ is a prime ideal of $R_P$, and so the fact that $xy \in I_P \subseteq \bigcap_{n \in \mathbb{N}} Q_P^n$ implies that either $x$ or $y$ belongs to $\bigcap_{n \in \mathbb{N}} Q_P^n$. Suppose, without loss of generality, that $x \in Q_P^n$ for every $n \in \mathbb{N}$. Then for each $n \in \mathbb{N}$, we can take $s_n \notin R \setminus P$ such that $s_n x \in Q^n$ and, since $Q^n$ is $P$-primary (by part (1)) and $s_n \notin \operatorname{Rad} Q_n$, we see that $x \in Q^n$. Hence $x \in I$, and we can conclude that $I$ is prime.

(3) By part (1), $P$ and $P^2$ are distinct $P$-primary ideals of $R$ and, therefore, $P_P$ and $P_P^2$ are distinct $P_P$-primary ideals of $R_P$, that is, $P_P$ is not an idempotent ideal of $R_P$. Since $R_P$ is a valuation domain, the $P_P$-primary ideals of $R_P$ are precisely the powers of $P_P$. Since the $P_P$-primary ideals of $R_P$ are in one-to-one correspondence with the $P$-primary ideals of $R$, we conclude that the $P$-primary ideals of $R$ are precisely the powers of $P$.

(4) Exercise. □

**Overrings of Prüfer Domains.** Overrings of Prüfer domains are Prüfer domains, and every Prüfer domain is integrally closed. In this section we argue these two facts and use them to give further characterizations of a Prüfer domain.

**Proposition 4.** *Every Prüfer domain is integrally closed.*

*Proof.* Let $R$ be a Prüfer domain, and let $q$ be a nonzero element in $\operatorname{qf}(R)$ that is integral over $R$. The $R$-submodule $M := R[q]$ is finitely generated. Therefore it is a fractional ideal of $R$, and so there is an $r \in R$ such that $rM$ is a finitely generated ideal of $R$. As $R$ is a Prüfer domain, there is a fractional ideal $J$ such that $(rJ)M = J(rM) = R$. Hence $M$ is an invertible fractional ideal. This, along with the fact that $M^2 = M$, implies that $M \subseteq R$. As a result, $q \in R$. Hence we conclude that $R$ is integrally closed. □

Our next goal is to study the overrings of a Prüfer domain.

**Proposition 5.** *Let $R$ be an Prüfer domain, and let $V$ be a valuation overring of $R$. Then there exists a prime ideal of $R$ such that $V = R_P$.*

*Proof.* We have seen that every valuation domain is local. Let $M$ be the maximal ideal of $V$, and set $P = M \cap R$. To see that $R_P \subseteq V$, take $r/s \in R_P$ with $r \in R$ and $s \in R \setminus P$. Since $s \in R \setminus P$, then $s \notin M$. Therefore $s \in V^\times$, which implies that $r/s = s^{-1}r \in V$.

To argue the reverse inclusion assume, by way of contradiction, that there is a $v \in V$ such that $v \notin R_P$. Since $R$ is a Prüfer domain, $R_P$ is a valuation domain. This, along with the fact that $v \notin R_P$, ensures that $v^{-1} \in R_P$. Write $v^{-1} = r/s$ for some $r, s \in R$ such that $s \notin P$. As $v \notin R_P$, it follows that $r \in P \subseteq M$. Thus $s = rv \in M \cap R = P$, which is a contradiction. □

Before giving further characterization of a Prüfer domain, let us argue the following lemma.

**Lemma 6.** *Let $R$ be an integrally closed local domain, and let $u \in \mathrm{qf}(R)^\times$ be a root of a polynomial in $R[x]$ with at least one of the coefficients in $R^\times$. Then either $u \in R$ or $u^{-1} \in R$.*

*Proof.* We use induction on the degree the polynomials described in the statement of the lemma. The case when one of such polynomials has degree one follows easily. Assume that the existence of a polynomial of degree at most $n-1$ (for $n \geq 2$) with $u$ as a root and a coefficient in $R^\times$ guarantees that $u \in R$ or $u^{-1} \in R$. Now take $\sum_{i=0}^{n} c_n x^n \in R[x]$ to be an $n$-degree polynomial with $u$ as a root and a coefficient in $R^\times$. If $c_n \in R^\times$, then $u$ is integral over $R$ and, as $R$ is integrally closed, $u \in R$. Suppose, otherwise, that $c_n \notin R^\times$. Since $(c_n u)^n + \sum_{i=0}^{n-1} c_i c_n^{n-i-1} (c_n u)^i = 0$, it follows that $c_n u$ is integral over $R$ and so $c_n u \in R$. If $c_n u \in R^\times$, then $u^{-1} \in R$. Then we assume that $c_n u \notin R^\times$. Observe now that if $c_{n-1} \in R^\times$, then the fact that $R$ is local implies that $c_n u + c_{n-1} \in R^\times$, and so the equality $(c_n u + c_{n-1}) u^{n-1} + \sum_{i=0}^{n-2} c_i u^i = 0$ guarantees that $u \in R$. Finally, if $c_{n-1} \notin R^\times$, then $c_i \in R^\times$ for some $i \in [\![0, n-2]\!]$, and so the induction hypothesis applied to the polynomial $(c_n u + c_{n-1}) x^{n-1} + \sum_{i=0}^{n-2} c_i x^i$ ensures that either $u \in R$ or $u^{-1} \in R$. □

Now we can characterize a Prüfer domain in terms of its overrings.

**Theorem 7.** *For an integral domain $R$, the following statements are equivalent.*

(a) *$R$ is a Prüfer domain.*
(b) *Every overring of $R$ is a Prüfer domain.*
(c) *Every overring of $R$ is integrally closed.*

*Proof.* (a) ⇒ (b): Let $T$ be an overring of $R$, and let $M$ be a prime ideal of $T$. Set $P = M \cap R$. For $S := R \setminus P$, we observe that $R_P = S^{-1}R \subseteq S^{-1}T \subseteq T_M$. Since $R$ is a Prüfer domain, $R_P$ is a valuation domain. We have seen that the overrings of a valuation domain are in bijection with its prime ideals and can be obtained by localizing at them. Therefore $T_M$ is the localization of $R_P$ at some prime ideal. Since $R_P$ is a local domain, it follows that $T_M = R_P$. Since $R$ is a Prüfer domain, $T_M = R_P$ is a valuation domain. As $T_M$ is a valuation domain for every prime ideal $M$ of $T$, we conclude that $T$ is a Prüfer domain.

(b) $\Rightarrow$ (c): This is clear as we have seen in Proposition 4 that every Prüfer domain is integrally closed.

(c) $\Rightarrow$ (a): It suffices to prove that $R_M$ is a valuation domain for every maximal ideal $M$ of $R$. Let $M$ be a maximal ideal of $R$, and set $S := R_M$. Take $u \in \text{qf}(S)$. Since $S[u^2]$ is an overring of $R$, it must be integrally closed. As $u$ is integral over $S[u^2]$, it follows that $u \in S[u^2]$, and so $u = \sum_{i=0}^n c_i u^{2i}$ for some $c_0, \dots, c_n \in S$. Thus, $u$ is a root of the polynomial $x - \sum_{i=0}^n c_i x^{2i} \in S[x]$. Because $S$ is an integrally closed local domain and $1 \in S^\times$ is a coefficient of $p(x)$, Lemma 6 guarantees that either $u \in S$ or $u^{-1} \in S$. Hence $S = R_M$ is a valuation domain, which concludes the proof. □

**Corollary 8.** *Let $R$ be a Prüfer domain, and let $S$ be a submonoid of $R^*$. Then $S^{-1}R$ is a Prüfer domain.*

The following result can be used to construct Prüfer domains.

**Theorem 9.** *Let $R$ be a Prüfer domain, and let $F$ be an algebraic extension of* $\text{qf}(R)$. *Then the integral closure of $R$ in $F$ is a Prüfer domain.*

*Proof.* Let $S$ denote the integral closure of $R$ in $F$, and suppose that $M$ is a maximal ideal of $S$ and set $P = M \cap R$. Take $\alpha \in F$. As $F$ is an algebraic extension of $\text{qf}(R)$, we see that $\alpha$ is a root of a nonzero polynomial $p(x)$ in $R_P[x]$.Since $P$ is a prime ideal of the Prüfer domain $R$, the localization $R_P$ is a valuation domain. So after normalizing $p(x)$ we can assume that it has a coefficient that is a unit in $R_P$. Observe that $R_P \subseteq S_M$ and, therefore, $p(x)$ is a polynomial in $S_M[x]$ having a unit of $S_M$ as a coefficient. As $S_M$ is an integrally closed local domain, it follows from Lemma 6 that either $\alpha \in S_M$ or $\alpha^{-1} \in S_M$. Hence $S_M$ is a valuation of $F$. As the localization of $S$ at any maximal ideal is a valuation domain, $S$ must be a Prüfer domain. □

## Exercises

**Exercise 1.** *Let $R$ be a Prüfer domain, and let $P$ be a prime ideal of $R$. Prove that the intersection of all $P$-primary ideals of $R$ is a prime ideal that contains each prime ideal properly contained in $P$. [Hint: Use the similar statement already proved for valuation domains.]*

**Exercise 2.** *Let $R$ be an integral domain, and let $\{V_\gamma : \gamma \in \Gamma\}$ be the set consisting of all the valuation overrings of $R$. For an ideal $I$ of $R$, the set $I' := \bigcap_{\gamma \in \Gamma} IV_\gamma$ is called the* completion *of $I$, and $I$ is called* complete *if $I' = I$.*

(1) *Prove that the following statements are equivalent.*
   (a) *$R$ is integrally closed.*
   (b) *Every principal ideal of $R$ is complete.*
   (c) *There is a nonzero principal ideal of $R$ that is complete.*

(2) *Prove that $R$ is a Prüfer domain if and only if every ideal of $R$ is complete.*

**Exercise 3.** *Let $R$ be the direct limit of a directed system $(R_\gamma)_{\gamma \in \Gamma}$ of Prüfer domains. Prove that $R$ is also a Prüfer domain.*

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