IDEAL THEORY AND PRÜFER DOMAINS

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Prüfer Domains I

Throughout this lecture, R is an integral domain. Recall that $\mathrm{qf}(R)$ denotes the quotient field of R.

Definition and Examples. Prüfer domains, which are natural generalizations of valuation domains, play a fundamental role in multiplicative ideal theory. In this lecture, we start our discussion of Prüfer domains.

A fractional ideal I of R is *invertible* if there is a factional ideal J such that $IJ = R$, in which case $J = (R : I) = \{r \in \mathrm{qf}(R) : rI \subseteq R\}$. It is clear that the set of all invertible fractional ideals of R is an abelian group with identity R. Observe that such a group contains the set of all nonzero principal fractional ideals as a subgroup.

Definition 1. An integral domain R is a *Prüfer domain* if every nonzero finitely generated ideal of R is invertible.

Fields and PIDs are clearly Prüfer domains. Recall that a Bezout domain is an integral domain where every finitely generated ideal is principal. Since nonzero principal ideals are invertible, every Bezout domain is a Prüfer domain. In particular, every valuation domain is a Prüfer domain. Let us briefly exhibit two further examples of Prüfer domains.

Example 2. The set $\mathrm{Int}(R) := \{p(x) \in \mathbb{Q}[x] : p(\mathbb{Z}) \subseteq \mathbb{Z}\}$ is a subring of $\mathbb{Q}[x]$ called the *ring of integer-valued polynomials*. We shall prove soon enough that $\mathrm{Int}(R)$ is a non-Noetherian Prüfer domain of Krull dimension 2.

Example 3. The ring consisting of all the entire function on the complex plane is a Bezout domain of infinite Krull dimension. In particular, it is a Prüfer domain.

Although every PID is Prüfer, this is not the case for UFDs. The following example sheds some light upon this observation.

Example 4. For a field F, consider the ring of polynomials $R := F[x, y]$ and the ideal $I = Rx + Ry$ of R. If $f \in \mathrm{qf}(R)$ belongs to $J := (R : I)$, then $Rxf + Ryf \subseteq R$, and so $xf \in R$ and $yf \in R$. Therefore $f \in x^{-1}R \cap y^{-1}R = R$. Then $J \subseteq R$ (indeed, $J = R$), and we see that $IJ \subseteq I$. Thus, I is not an invertible ideal even though it is finitely generated, and this allows us to conclude that R is not a Prüfer domain. Note, however, that R is a UFD.

Characterizations. We will discuss various of the many characterizations of Prüfer domains. Let us start by the following.

Proposition 5. *For an integral domain R, the following statements are equivalent.*

(a) *R is a Prüfer domain.*

(b) *Every two-generated ideal of R is invertible.*

Proof. (a) $\Rightarrow$ (b): This is obvious.

(b) $\Rightarrow$ (a): We will show that every nonzero finitely generated ideal of R is invertible by using induction on the minimum number n of generators of such an ideal. It is clear when $n = 1$, and it follows from part (b) when $n = 2$. Suppose, therefore, that I can be generated by n elements, where $n > 2$, and assume that every nonzero ideal of R that can be generated by less than n elements is invertible. Now write $I = Rc_1 + \cdots + Rc_n$ for some nonzero elements $c_1, \dots, c_n \in R$. Set $I_1 := Rc_1 + \cdots + Rc_{n-1}$, $I_2 := Rc_2 + \cdots + Rc_n$, and $I_3 := Rc_1 + Rc_n$. By induction, I_1, I_2, and I_3 are invertible. Then $J := c_1 I_1^{-1} I_3^{-1} + c_n I_2^{-1} I_3^{-1}$ is a fractional ideal of R. We claim that J is the inverse of I. To show this, first observe that

$$\begin{aligned} IJ &= (I_1 + Rc_n) c_1 I_1^{-1} I_3^{-1} + (Rc_1 + I_2) c_n I_2^{-1} I_3^{-1} \\ &= c_1 I_3^{-1} + c_1 c_n I_1^{-1} I_3^{-1} + c_1 c_n I_2^{-1} I_3^{-1} + c_n I_3^{-1} \\ &= c_1 I_3^{-1}(R + c_n I_2^{-1}) + c_n I_3^{-1}(R + c_1 I_1^{-1}). \end{aligned}$$

As I_1 and I_2 are invertible ideals and $c_1 \in I_1$ and $c_n \in I_2$, it follows that $c_1 I_1^{-1} \subseteq R$ and $c_n I_2^{-1} \subseteq R$. This, along with the previous chain of equalities, guarantees that $IJ = c_1 I_3^{-1} + c_n I_3^{-1} = I_3 I_3^{-1} = R$. Hence I is an invertible ideal. $\square$

We proceed to characterize Prüfer domains in terms of valuation domains.

Proposition 6. *For an integral domain R, the following statements are equivalent.*

(a) *R is a Prüfer domain.*

(b) *R_P is a valuation domain for every prime ideal P.*

(c) *R_M is a valuation domain for every maximal ideal M.*

Proof. (a) $\Rightarrow$ (b): Assume that R is a Prüfer domain, and let P be a prime ideal of R. Since R_P is a local ring, it is enough to prove that it is a Bezout domain. Let $\frac{a_1}{s_1} R_P + \cdots + \frac{a_k}{s_k} R_P$ be a nonzero finitely generated ideal of R_P, where $a_1, \dots, a_k \in I$ and $s_1, \dots, s_k \in R \setminus P$. Then $I := Ra_1 + \cdots + Ra_k$ satisfies that $I_P = \frac{a_1}{s_1} R_P + \cdots + \frac{a_k}{s_k} R_P$. Since R is a Prüfer domain, I is invertible. Let J be a fractional ideal such that $JI = R$, then $(JR_P) I_P = R_P$, and so I_P is invertible in R_P. Since R_P is local, I_P is a principal ideal. Hence R_P is a valuation domain.

(b) $\Rightarrow$ (c): This is clear.

(c) $\Rightarrow$ (a): Assume, by way of contradiction, that there is a nonzero finitely generated ideal I of R that is not invertible. Write $I = Ra_1 + \cdots + Ra_n$ for $a_1, \dots, a_n \in R$. Since I is not invertible, $IJ \subsetneq R$, where $J := (R : I)$. So there is a maximal ideal M of R such that $IJ \subseteq M$. Because the extension I_M of I is a finitely generated ideal of the Bezout domain R_M, there is an $a \in I$ satisfying $I_M = aR_M$. For each $i \in [\![1, n]\!]$, we can now take $s_i \in R \setminus M$ with $s_i a_i \in aR$. After setting $s = s_1 \cdots s_n$, we see that $sa^{-1}a_i \in R$ for every $i \in [\![1, n]\!]$, and so $sa^{-1}I \subseteq R$. This implies that $sa^{-1} \in J$ and, therefore, $s = a(sa^{-1}) \in IJ \subseteq M$, which is a contradiction. □

Corollary 7. *Let R be a Prüfer domain, and let P be a prime ideal of R. Then the set of all P-primary ideals of R is totally ordered, and the intersection P' of all such primary ideals is a prime ideal satisfying that there is no prime ideal strictly between P' and P*

Proof. It follows from Proposition 6 that R_P is a valuation domain. Now the corollary follows from the correspondence between the P-primary ideals of R and the P_P-primary ideals of R_P, as we have seen before that the statement of the corollary holds for valuation domains. □

Prüfer domains can also be characterized using cancellation of finitely generated ideals.

Proposition 8. *For an integral domain R, the following statements are equivalent.*

(a) *R is a Prüfer domain.*

(b) *For every nonzero finitely generated ideal I of R, whenever $IB = IC$ for ideals B and C the equality $B = C$ must hold.*

(c) *For every finitely generated ideal I of R, whenever an ideal J is contained in I, there is an ideal K such that $J = IK$.*

Proof. (a) $\Rightarrow$ (b): Let I be a finitely generated nonzero ideal of R, and let J and K be ideals of R such that $IJ = IK$. Since R is a Prüfer domain, I is invertible and so $J = I^{-1}IJ = I^{-1}IK = K$.

(b) $\Rightarrow$ (a): Suppose, on the other hand, that every finitely generated nonzero ideal of R is cancellative. We start by observing that if I is a nonzero finitely generated ideal of R and $IJ \subseteq IK$ for ideals J and K of R, then $J \subseteq K$. Indeed, $IK = IJ + IK = I(J + K)$ implies that $K = J + K$, which means that $J \subseteq K$.

To prove that R is Prüfer it suffices to argue that the localization of R at any prime ideal is a valuation domain. Let P be a prime ideal of R. Take $a, b \in R$, and let us show that either $aR_P \subseteq bR_P$ or $bR_P \subseteq aR_P$. The assertion clearly holds when $a = 0$ or $b = 0$. So we assume that $ab \neq 0$. Note that $Rab(Ra + Rb) \subseteq (Ra^2 + Rb^2)(Ra + Rb)$, and so $Rab \subseteq Ra^2 + Rb^2$. Take $x, y \in R$ such that $ab = xa^2 + yb^2$, and observe that $Ryb(Ra + Rb) \subseteq Ra(Ra + Rb)$. Therefore $yb = ra$ for some $r \in R$, and we can write

$ab = xa^2 + rab$, that is, $xa = b(1-r)$. If $r \notin P$, then $a = b(y/r) \in bR_P$ and so $aR_P \subseteq bR_P$. On the other hand, if $r \in P$, then $1-r \notin P$ and so $b = ax/(1-r) \in aR_P$, which implies that $bR_P \subseteq aR_P$. Hence R_P is a valuation domain for every prime ideal P.

(a) $\Rightarrow$ (c): Let I be a finitely generated ideal of R, and let J be an ideal of R contained in I. If I is the zero ideal so is J, and we can take $K = R$ (or any ideal of R). If I is nonzero, then it is invertible and so we can take K to be $I^{-1}J$.

(c) $\Rightarrow$ (a): Finally, suppose that the statement (c) holds. We will show that every localization of R at a prime ideal is a valuation. To do so, take a prime ideal P of R. Take $a, b \in R$ and let us verify that the principal ideals aR_P and bR_P are comparable. Since $Ra \subseteq Ra + Rb$, there is an ideal I such that $Ra = (Ra + Rb)I$. After writing $a = xa + yb$ for some $x, y \in I$, we see that $yb = a(1-x) \in aR$. If $x \in P$, then $1-x \notin P$, and from $a = by/(1-x) \in bR_P$ we obtain that $aR_P \subseteq bR_P$. On the other hand, if $x \notin P$, then $bx \in bI \subseteq (Ra + Rb)I = Ra$ ensures that $b \in aR_P$, that is, $bR_P \subseteq aR_P$. Hence R_P is a valuation domain for every prime ideal P. □

Finally, we characterize Prüfer domains by using certain distributivity laws.

Proposition 9. *For an integral domain R, the following statements are equivalent.*

(a) *R is a Prüfer domain.*

(b) *$A(B \cap C) = AB \cap AC$ for all ideals A, B, and C of R.*

(c) *$A \cap (B + C) = A \cap B + A \cap C$ for all ideals A, B, and C of R.*

Proof. (a) $\Rightarrow$ (b): Suppose that R is a Prüfer domain, and let A, B, and C be ideals of R. Let P be a maximal ideal of R. Since R_P is a valuation domain by Proposition 6, the ideals BR_P and CR_P of R_P are comparable and, therefore,

$$A(B \cap C)R_P = AR_P(BR_P \cap CR_P) = (AR_P)(BR_P) \cap (AR_P)(CR_P) = (AB \cap AC)R_P.$$

Since the maximal ideal P was arbitrarily taken, the equality $A(B \cap C) = AB \cap AC$ must hold.

(b) $\Rightarrow$ (a): Take $C = Rc_1 + Rc_2$ for some $c_1, c_2 \in R$, and let us check that C is an invertible ideal. This is clear if C is principal. Therefore suppose that $c_1 \neq 0$ and $c_2 \neq 0$. Set $A = Rc_1$ and $B := Rc_2$, and observe that $AB \subseteq CA \cap CB = C(A \cap B)$, and so $AB = (A \cap B)C$. As both A and B are invertible ideals,

$$C(A \cap B)B^{-1}A^{-1} = ABB^{-1}A^{-1} = R,$$

which implies that C is also an invertible ideal. Since every two-generated ideal of R is invertible, it follows from Proposition 5 that R is a Prüfer domain.

(a) $\Rightarrow$ (c): This follows the same argument used to establish (a) $\Rightarrow$ (b).

(c) $\Rightarrow$ (a): Fix a prime ideal P, and let us verify that R_P is a valuation domain. To do so, take $a, b \in R$ and observe that, in light of the distributive law in (c),

$$Ra = Ra \cap (Rb + R(a-b)) = (Ra \cap Rb) + (Ra \cap R(a-b)).$$

As a consequence, one can pick $t \in Ra \cap Rb$ and $r \in R$ with $r(a-b) \in Ra$ such that $a = t + r(a-b)$. Then we see that $rb \in Ra$ and $(1-r)a \in Rb$. Thus, if $r \in P$, then $1 - r \notin P$, which implies that $a \in bR_P$. On the other hand, if $r \notin P$, then $a - b \in aR_P$ and so $b \in aR_P$. Therefore the ideals aR_P and bR_P are comparable. Because any two principal ideals of R_P are comparable, R_P is a valuation domain, and it follows from Proposition 6 that R is a Prüfer domain. □

Exercises

Exercise 1. *Let R be a Prüfer domain, and let P be a prime ideal of R. Prove that R/P is also a Prüfer domain.*

Exercise 2. *Let R be an integral domain. Prove that the following statements are equivalent.*

(1) *R is a Prüfer domain.*
(2) *$(J+K) : I = (J : I) + (K : I)$ for all ideals $I, J,$ and K of R with I finitely generated.*
(3) *$I : (J \cap K) = (I : J) + (I : K)$ for all ideals $I, J,$ and K of R with J and K finitely generated.*

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