# IDEAL THEORY AND PRÜFER DOMAINS

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## Valuation Domains II

Throughout this lecture, $R$ is an integral domain. Recall that $\mathrm{qf}(R)$ denotes the quotient field of $R$.

**Overrings and Underrings.** An extension ring $S$ of $R$ is called an *overring* if $S$ is a subring of $\mathrm{qf}(R)$, in which case, $\mathrm{qf}(S) = \mathrm{qf}(R)$. It is clear that every overring of a valuation domain is a valuation domain. A subring $U$ of $R$ is called an *underring* of $R$ if $\mathrm{qf}(U) = \mathrm{qf}(R)$, in which case, $R$ is an overring of $U$. Given a valuation domain, one can obtain all its valuation overrings (resp., underrings) by looking at localizations (resp., quotients). We prove this in the next two propositions.

**Proposition 1.** *Let $R$ be a valuation domain. Then the overrings of $R$ are in bijection with the prime ideals of $R$ and can be obtained by localization at prime ideals.*

*Proof.* Let $M_R$ denote the maximal ideal of $R$. Let $\mathscr{O}$ be the set consisting of all overrings of $R$, and let $\mathscr{P}$ be the set consisting of all prime ideals of $R$. Take $S$ to be an overring of $R$ with maximal ideal $M_S$. Since $M_S$ contains no units of $S$, for each $s \in M_S$, the element $s^{-1} \notin R$ and, therefore, the fact that $R$ is a valuation domain ensures that $s \in R$. Hence $M_S \subseteq R$. As $M_S$ is a prime ideal in $S$, it must be a prime ideal in $R$. Hence the assignment $S \mapsto M_S$ induces a map $\varphi\colon \mathscr{O} \to \mathscr{P}$. For every prime ideal $P \in \mathscr{P}$, the localization $R_P$ is an overring of $R$ satisfying $\varphi(R_P) = P$. Thus, $\varphi$ is surjective. To check that $\varphi$ is injective, take two overrings $S_1$ and $S_2$ of $R$ such that $\varphi(S_1) = M = \varphi(S_2)$. If $s \in S_1 \setminus M = S_1^\times$, then either $s$ or $s^{-1}$ belongs to $S_2 \setminus M$ because $S_2$ is a valuation domain. Then the equality $S_2^\times = S_2 \setminus M$ implies that $s \in S_2 \setminus M$. As a result, $S_1 \subseteq S_2$, and we can similarly check the reverse inclusion. Hence $\varphi$ is also injective, which concludes the proof. □

**Proposition 2.** *Let $R$ be a valuation domain with maximal ideal $M_R$. Then the valuation underrings of $R$ are in bijection with the valuation subrings of $R/M_R$.*

*Proof.* Suppose that $U$ is a valuation underring of $R$. Since $R$ is an overring of $U$, it follows that $M_R$ is a prime ideal of $U$ (see the proof of Proposition 1). As $\mathrm{qf}(U) = \mathrm{qf}(R)$, we can easily verify that $R/M_R$ is the quotient field of $U/M_R$. Now take $r \in R$ such that $r + M_R$ does not belong to $U/M_R$. Then $r \notin U$, and the fact that $U$ is a valuation domain implies that $r^{-1} \in U$, whence $r^{-1} + M_R \in U/M_R$. Thus, $U/M_R$ is a valuation domain. As a result, the assignment $U \mapsto U/M_R$ determines a map from the set of underrings of $R$ to the set of valuation subrings of $R/M_R$. This map is injective by the Third Isomorphism Theorem for rings. To show that it is surjective, suppose that $V/M_R$ is a valuation subring of $R/M_R$ for some subring $V$ of $R$. Take $r \in \mathrm{qf}(R)$ such that $r \in R$. Then either $r \in M_R \subseteq V$ or $r \in R^\times$. In the second case, either $r + M_R = v + M_R$ or $r^{-1} + M_R = v + M_R$ for some $v \in V$, and so either $r \in V$ or $r^{-1} \in V$. This argument, along with the fact that $R$ is a valuation domain, shows that $V$ is a valuation underring of $R$, which concludes the proof. ☐

**Integral closures.** Our next goal will be to show that the integral closure of an integral domain is the intersection of all its valuation overrings. Before proving this, we establish some useful results about valuation domains.

**Proposition 3.** *Let $R$ be a subring of a field $F$. For every prime ideal $P$ of $R$, there is a valuation domain of $F$ containing $R$ whose maximal ideal lies over $P$.*

*Proof.* After replacing $R$ by its localization at the prime ideal $P$ if necessary, one can assume that $R$ is a local ring with maximal ideal $P$. Let $\mathscr{S}$ be the poset consisting of all subrings $S$ of $F$ containing $R$ and satisfying that $1 \notin PS$. Clearly, $\mathscr{S}$ contains $R$. In addition, the union of all the subrings in any chain of $\mathscr{S}$ is again a subring of $F$ in $\mathscr{S}$. Hence $\mathscr{S}$ contains a maximal element $S$ by virtue of Zorn's Lemma. Let $M$ be a maximal ideal of $S$ containing the proper ideal $PS$. Since $R \subseteq S_M$ and $S_M \in \mathscr{S}$, the maximality of $S$ ensures that $S = S_M$. Thus, $S$ is a local ring. Since $M \cap R$ is a proper ideal of $R$ containing the maximal ideal $P$, it follows that $M$ lies over $P$.

To show that $S$ is a valuation domain of $F$, take $x \in F$ such that $x \notin S$. The maximality of $S$ ensures that $1 \in PS[x]$. Take $b_0, \dots, b_k \in PS$ such that $1 = \sum_{i=0}^k b_i x^i$. Since $b_0$ belongs to the only maximal ideal of $S$, it follows that $1 - b_0 \in S^\times$. As a result, there is a minimum $m \in \mathbb{N}$ such that there exist $c_1, \dots, c_m \in M$ with

$$1 = c_1 x + \cdots + c_m x^m. \tag{0.1}$$

Now suppose, by way of contradiction, that $x^{-1} \notin S$. Mimicking the previous argument, we can guarantee the existence of a minimum $n \in \mathbb{N}$ such that

$$1 = c'_1 x^{-1} + \cdots + c'_n x^{-n} \tag{0.2}$$

for some $c'_1, \dots, c'_n \in M$. Observe that if $m \ge n$, then we can add the equation (0.2) multiplied by $c_m x^m$ to the equation (0.1) to contradict the minimality of $m$. On the other hand, if $m < n$, then we can add the equation (0.1) multiplied by $c'_n x^{-n}$ to the equation (0.2) to contradict the minimality of $n$. Hence $x^{-1} \in S$. As a result, $S$ is a valuation domain of $F$. ☐

Valuation domains are integrally closed, as the following proposition shows.

**Proposition 4.** *Every valuation domain is integrally closed.*

*Proof.* Let $q \in \mathrm{qf}(R)^\times$ be an integral element over $R$, and take a polynomial $x^n - \sum_{i=0}^{n-1} c_i x^i$ in $R[x]$ having $q$ as a root. If $q^{-1} \in R$, then $q^{-n} = \sum_{i=0}^{n-1} c_i q^{-i}$, and so $1 = q^{-1}\big(\sum_{i=0}^{n-1} c_i q^{n-i+1}\big)$. In this case, $q^{-1} \in R^\times$ and, therefore, $q \in R$. On the other hand, if $q^{-1} \notin R$, then $q \in R$ because $R$ is a valuation domain. Thus, $R$ is integrally closed. $\square$

We are in a position to prove the main result of this lecture.

**Theorem 5.** *Let $R$ be a subring of a field $F$. Then the integral closure of $R$ in $F$ equals the intersection of all the valuation domains of $F$ containing $R$.*

*Proof.* Let $\bar{R}$ denote the integral closure of $R$ in $F$. Since every valuation domain is integrally closed, it is clear that $\bar{R}$ is contained in the intersection of all valuation domains of $F$ containing $R$. For the reverse implication, suppose that $x \in F^\times$ is not integral over $R$, and set $y = x^{-1}$. We claim that $yR[y] \neq R[y]$. If this were not the case, then $1 = \sum_{i=1}^n c_i y^i$ for some $c_1, \dots, c_n \in R$, whence $x^n - \sum_{i=1}^n c_i x^{n-i} = 0$, contradicting that $x$ is not integral over $R$. Therefore the claim follows, that is, $yR[y]$ is a proper ideal of $R[y]$. Let $M$ be a maximal ideal of $R[y]$ containing $yR[y]$. Proposition 3 now guarantees the existence of a valuation domain $S$ of $F$ with its maximal ideal $M_S$ satisfying $M_S \cap R = M$. Since $y \in M_S$, the valuation domain $S$ does not contain $x$. Hence the intersection of all the valuation domains of $F$ containing $R$ is contained in $\bar{R}$. $\square$

**Corollary 6.** *The integral closure of an integral domain is the intersection of all its valuation overrings.*

**Homomorphism Extensions.** For fields $F$ and $K$ and a subring $R$ of $F$, we are interested in whether we can extend a ring homomorphism $\varphi\colon R \to K$ to a larger subring of $F$. If so, we would like to know how much $\varphi$ can be extended. The following lemma gives a plausible answer to our first concern. In addition, Theorem 8 can be taken as an effective answer to our second concern.

**Lemma 7.** *Let $F$ be a field, $R$ a subring of $F$, and $\varphi\colon R \to K$ a ring homomorphism, where $K$ is an algebraically closed field. If $\alpha \in F^\times$, then $\varphi$ can be extended to either a ring homomorphism $R[\alpha] \to K$ or a ring homomorphism $R[\alpha^{-1}] \to K$.*

*Proof.* Letting $P$ denote the kernel of $\varphi$, we can extend $\varphi$ to a ring homomorphism $R_P \to K$ via the assignment $r/s \mapsto \varphi(r)/\varphi(s)$ for every $r \in R$ and $s \in R \setminus P$. Note that the kernel of the extended homomorphism is the maximal ideal $P_P$, and so its image is a subfield of $K$ isomorphic to $R_P/P_P$. Therefore we can assume, without loss of generality, that $R$ is a local ring and $\varphi(R)$ is a subfield of $K$. Let $M$ denote the maximal ideal of $R$. We can extend $\varphi\colon R \to K$ to a ring homomorphism $\varphi_x\colon R[x] \to \varphi(R)[x]$ via the natural assignments $\sum_{i=0}^n r_i x^i \mapsto \sum_{i=0}^n \varphi(r_i) x^i$. Set $I_\alpha := \{p(x) \in R[x] : p(\alpha) = 0\}$

and $I_{\alpha^{-1}} := \{p(x) \in R[x] : p(\alpha^{-1}) = 0\}$. As $\varphi_x$ is surjective, $J_\alpha := \varphi(I_\alpha)$ and $J_{\alpha^{-1}} := \varphi(I_{\alpha^{-1}})$ are ideals of $\varphi(R)[x]$. We claim that at least one of these ideals must be proper.

Suppose, by way of contradiction, that $J_\alpha = J_{\alpha^{-1}} = \varphi(R)[x]$. Take a polynomial $f(x) = \sum_{i=0}^k c_i x^i \in I_\alpha$ with minimum degree such that $\varphi_x(f(x)) = 1$. Similarly, take $g(x) = \sum_{i=0}^\ell d_i x^i \in I_{\alpha^{-1}}$ with minimum degree such that $\varphi_x(g(x)) = 1$. Assume first that $k \ge \ell$. Because $\varphi(d_0) = \varphi_x(g(0)) = 1$, it follows that $1 - d_0 \in \ker \varphi \subseteq M$ and, therefore, $d_0 \notin M$. As $R$ is a local ring, $d_0 \in R^\times$. Now we can subtract from $\sum_{i=0}^k c_i \alpha^i = 0$ the equality $\sum_{i=0}^\ell d_i \alpha^{-i} = 0$ multiplied by $d_0^{-1} c_k \alpha^k$ to contradict the minimality of $f(x)$. We can arrive to a contradiction in a similar way under the assumption that $\ell \ge k$.

Thus, at least one of $J_\alpha$ and $J_{\alpha^{-1}}$ is a proper ideal of $\varphi(R)[x]$. Suppose, without loss of generality, that $J_\alpha$ is proper. Since $\varphi(R)$ is a field, $\varphi(R)[x]$ is a PID and, therefore, $J_\alpha$ is a principal ideal. Write $J_\alpha = (q(x))$ for some $q(x) \in \varphi(R)[x]$. As $J_\alpha$ is proper, we see that $q(x) \notin \varphi(R)^\times$. Because $K$ is algebraically closed, $q(x)$ must have a root $\rho$ in $K$. Define $\bar\varphi \colon R[\alpha] \to K$ via $\sum_{i=0}^n r_i \alpha^i \mapsto \sum_{i=0}^n \varphi(r_i)\rho^i$. To see that $\bar\varphi$ is well defined, take $p(x) \in R[x]$ with $p(\alpha) = 0$; that is, $p(x) \in I_\alpha$. Then $\varphi_x(p(x)) \in J_\alpha$, and so it is a multiple of $q(x)$, which implies that $\varphi_x(p(x))$ has $\rho$ as a root. Hence $\bar\varphi$ is the desired extension of $\varphi$. $\square$

**Theorem 8.** *Let $F$ be a field, $R$ a subring of $F$, and $\varphi \colon R \to K$ a ring homomorphism, where $K$ is an algebraically closed field. Then the following statements hold.*

1. *There is a maximal extension $\bar\varphi \colon V \to K$ of $\varphi$ inside $F$.*
2. *If $\bar\varphi \colon V \to K$ is a maximal extension of $\varphi$ inside $F$, then $V$ is a valuation of $F$.*

*Proof.* (1) Consider the set $\mathscr{P}$ consisting of all pairs $(S, \sigma)$, where $S$ is a subring of $F$ containing $R$ and $\sigma \colon S \to K$ is a ring homomorphism extending $\varphi$. Define $\le$ on $\mathscr{P}$ as follows: $(S_1, \sigma_1) \le (S_2, \sigma_2)$ whenever $S_1 \subseteq S_2$ and $\sigma_2$ is an extension of $\sigma_1$. Clearly, $\mathscr{P}$ is a nonempty poset. Now suppose that $\mathscr{T} := \{(S_i, \sigma_i) : i \in I\}$ is a nonempty totally ordered subset of $\mathscr{P}$. As $\mathscr{T}$ is totally ordered, $S := \bigcup_{i \in I} S_i$ is a subring of $F$ containing $R$. Define $\sigma \colon S \to K$ by $\sigma(s) = \sigma_i(s)$ choosing $i \in I$ so that $s \in S_i$. Since $\mathscr{T}$ is totally ordered, $\sigma$ is a ring homomorphism and, therefore, $(S, \sigma)$ is an upper bound for $\mathscr{T}$. Thus, Zorn's lemma guarantees the existence of a maximal extension $\bar\varphi \colon V \to K$ of $\varphi$.

(2) Let $\bar\varphi \colon V \to K$ be a maximal extension of $\varphi$ inside $F$. To check that $V$ is a valuation of $F$, take $\alpha \in F^\times$. It follows from Lemma 7 that $\bar\varphi$ can be extended to either a homomorphism $V[\alpha] \to K$ or a homomorphism $V[\alpha^{-1}] \to K$. The maximality of $(V, \bar\varphi)$ now ensures that either $\alpha \in V$ or $\alpha^{-1} \in V$. Hence $V$ is a valuation of $F$. $\square$

## Exercises

**Exercise 1.** *Let $R$ be a subring of a field $F$, and let $P_1 \subsetneq P_2 \subsetneq \cdots \subsetneq P_n$ be a chain of prime ideals of $R$. Show that there exists a valuation domain of $F$ containing $R$ and having prime ideals $Q_1, \dots, Q_n$ such that $Q_k$ lies over $P_k$ for every $k \in [\![1, n]\!]$.*

**Exercise 2.** *Derive Theorem 8 from Theorem 5.*

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