# IDEAL THEORY AND PRÜFER DOMAINS

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## Valuation Domains I

For an integral domain $R$, we let $\mathrm{qf}(R)$ denote the quotient field of $R$.

**Characterizations.** The primary purpose of this subsection is to introduce valuation domains and provide several useful characterizations.

**Definition 1.** An integral domain $R$ is called a *valuation domain* if for every nonzero $x \in \mathrm{qf}(R)$, either $x$ or $x^{-1}$ belongs to $R$.

We observe that if $R$ is a valuation domain, then for all nonzero $x, y \in R$ either $xy^{-1} \in R$ or $yx^{-1} \in R$, which means that $y \mid_R x$ or $x \mid_R y$.

It is clear from the definition that every field is a valuation domain. In addition, if $R$ is a valuation domain and $S$ is an extension ring of $R$ with $R \subseteq S \subseteq \mathrm{qf}(R)$, then $S$ is also a valuation domain. Let us take a look at another example.

**Example 2.** For $p \in \mathbb{P}$, and consider the localization $\mathbb{Z}_{(p)}$ of $\mathbb{Z}$ at the prime ideal $(p)$. The quotient field of $\mathbb{Z}_{(p)}$ is $\mathbb{Q}$. It is clear that for every nonzero rational number $q$, either $q \in \mathbb{Z}_{(p)}$ or $q^{-1} \in \mathbb{Z}_{(p)}$. As a result, $\mathbb{Z}_{(p)}$ is a valuation domain.

**Example 3.** What are the valuation domains of $\mathbb{C}(x)$? As in the previous example, we can readily see that $\mathbb{C}[x]_{(x-\alpha)}$ for any $\alpha \in \mathbb{C}$ are valuation domains of $\mathbb{C}(x)$. Indeed, these are the only valuation rings of $\mathbb{C}(x)$ containing $\mathbb{C}[x]$ (prove this!). However, there is a valuation domain of $\mathbb{C}(x)$ that does not contain $\mathbb{C}[x]$: the integral domain $R := \mathbb{C}[x^{-1}]_{(x^{-1})}$. To show that $R$ is a valuation domain it suffices to observe that $x \mapsto x^{-1}$ induces an automorphism of $\mathbb{C}(x)$ sending $\mathbb{C}[x]_{(x)}$ to $R$.

We can characterize a valuation domain in terms of its poset of (principal) ideals.

**Proposition 4.** *For an integral domain $R$, the following statements are equivalent.*

(a) *$R$ is a valuation domain.*
(b) *The principal ideals of $R$ are totally ordered by inclusion.*
(c) *The ideals of $R$ are totally ordered by inclusion.*
(d) *The principal fractional ideals of $R$ are totally ordered by inclusion.*
(e) *The fractional ideals of $R$ are totally ordered by inclusion.*

*Proof.* (a) $\Leftrightarrow$ (b): For all $x, y \in R$, it is clear that $x \mid_R y$ if and only if $yR \subseteq xR$, whence the equivalence follows.

(b) $\Leftrightarrow$ (c): Clearly, (c) implies (b). For the direct implication, suppose that $I_1$ and $I_2$ are ideals of $R$ such that $I_1 \nsubseteq I_2$. Take $x \in I_1 \setminus I_2$. Now take $y \in I_2$, and note that $y \nmid_R x$, which means that $xR \nsubseteq yR$. Since the principal ideals of $R$ are totally ordered, $y \in yR \subseteq xR \subseteq I_1$. Thus, $I_2 \subseteq I_1$.

(c) $\Rightarrow$ [(d) and (e)]: It suffices to check that (c) implies (e). Let $J_1$ and $J_2$ be two fractional ideals of $R$. Take $r_1, r_2 \in R$ such that $r_1 J_1$ and $r_2 J_2$ are ideals of $R$. Then $r_1 r_2 J_1$ and $r_1 r_2 J_2$ are also ideals of $R$, and so they must be comparable. Hence the fractional ideals $J_1$ and $J_2$ are also comparable.

[(d) or (e)] $\Rightarrow$ (b): This is obvious. □

**Corollary 5.** *Every valuation domain is a local ring.*

*Proof.* It follows from Proposition 4 that the ideals of $R$ are totally ordered by inclusion. Therefore $R$ must contain exactly one maximal ideal, namely, the union of all proper ideals. □

An integral domain $R$ is called a *Bezout domain* if every finitely generated ideal of $R$ is principal. As the following proposition illustrates, valuation domains can also be characterized as local Bezout domains.

**Proposition 6.** *An integral domain is a valuation domain if and only if it is a local Bezout domain.*

*Proof.* For the direct implication, suppose that $R$ is a valuation domain. Then $R$ is local by Corollary 5. To show that $R$ is a Bezout domain, let $I = (r_1, \dots, r_n)$ be a finitely generated ideal of $R$. Since $R$ is a valuation domain, the set $\{r_j R : j \in [\![1, n]\!]\}$ is totally ordered under inclusion, whence it has a maximum element, namely, $rR$ for some $r \in \{r_1, \dots, r_n\}$. It is clear that $I = rR$. Hence $R$ is a local Bezout domain.

For the reverse implication, suppose that $R$ is a local Bezout domain. Take $a, b \in R$, and consider the ideal $I := Ra + Rb$. Let $M$ be the maximal ideal of $R$. One can easily verify that the abelian group $I/MI$ is indeed an $R/M$-module, that is, a vector space over the field $R/M$. Since $I$ is finitely generated, it is principal and, therefore, the vector space $I/MI$ has dimension one. So there exist $u, v \in R$ such that $ua + vb \in MI$, where either $u \in R^\times$ or $v \in R^\times$. So we can take $r, s \in M$ such that $ua + vb = ra + sb$, that is, $(u - r)a = (s - v)b$. Since $u - r \in R^\times$ or $s - v \in R^\times$, it follows that either $b \mid_R a$ or $a \mid_R b$. Hence $R$ is a valuation domain. □

**Corollary 7.** *Every Noetherian valuation domain is a PID.*

A pair $(G, \leq)$, where $G$ is an additive abelian group and $\leq$ is an order relation on $G$, is called an *ordered group* provided that $\leq$ is translation-invariant, that is, for all $a, b, c \in G$, the inequality $b \leq c$ implies that $a + b \leq a + c$. If $\leq$ is a total order, then we say that $(G, \leq)$ is a *totally ordered group* or a *linearly ordered group*. To easy notation we often write $G$ instead of $(G, \leq)$. For an ordered group $G$, the set $G_+ := \{a \in G : a \geq 0\}$ is a submonoid of $G$, which is called the *nonnegative cone* of $G$. If $G$ is a totally ordered group, then it follows immediately that, for every $a \in G \setminus \{0\}$, exactly one of the inclusions $a \in G_+$ and $-a \in G_+$ holds.

For a field $F$ and a totally ordered (abelian) group $G$, a map $v \colon F \to G \cup \{\infty\}$, where $\infty$ is a symbol not in $G$ such that $x \leq \infty$ and $x + \infty = \infty$ for all $x \in G$, is called a *valuation map* if the following conditions hold:

(1) $v(0) = \infty$,
(2) $v \colon F^\times \to G$ is a group homomorphism, and
(3) $v(x + y) \geq \min\{v(x), v(y)\}$ for all $x, y \in F$.

Here is the characterization of a valuation domain that suggests the chosen terminology.

**Theorem 8.** *For an integral domain $R$, the following statements are equivalent.*

(a) *$R$ is a valuation domain.*
(b) *There exists a valuation map $v \colon \mathrm{qf}(R) \to G \cup \{\infty\}$ such that*
$$R = \{x \in \mathrm{qf}(R) : v(x) \geq 0\}.$$

*Also, if $v$ and $R$ are as in part (b), then the maximal of $R$ is $\{x \in \mathrm{qf}(R) : v(x) > 0\}$.*

*Proof.* (a) $\Rightarrow$ (b): Assume that $R$ is a valuation domain, and set $G := \mathrm{qf}(R)^\times / R^\times$. We will write $G$ additively, that is, $xR^\times + yR^\times = xyR^\times$ for all $x, y \in \mathrm{qf}(R)^\times$. Now define the binary relation $\leq$ on $G$ as follows: $xR^\times \leq yR^\times$ if $yx^{-1} \in R$. Using that $yx^{-1} \in R$ if and only if $Ry \subseteq Rx$ for all $x, y \in \mathrm{qf}(R)^\times$, one can readily check that $\leq$ is an order relation on $G$ compatible with the addition, and it follows from Proposition 4 that $G$ is indeed a totally ordered group. Also, we see that $G_+ = \{xR^\times : x \in R \setminus \{0\}\}$.

Now we can define $v \colon \mathrm{qf}(R) \to G \cup \{\infty\}$ by $v(0) = \infty$ and $v(x) = xR^\times$ if $x \neq 0$. It is clear that $v \colon \mathrm{qf}(R)^\times \to G$ is a group homomorphism. Observe that the inequality $v(x+y) \geq \min\{v(x), v(y)\}$ follows immediately when at least one of the elements $x$, $y$, and $x+y$ equals zero. Now take $x, y \in \mathrm{qf}(R)^\times$ with $x+y \neq 0$ and assume, without loss of generality, that $xR^\times \leq yR^\times$. Since $1 + y/x \in R$, the element $(1 + y/x)R^\times$ belongs to $G_+$ and, therefore,
$$v(x + y) = x(1 + y/x)R^\times = xR^\times + (1 + y/x)R^\times \geq xR^\times = v(x).$$
Hence $v(x + y) \geq \min\{v(x), v(y)\}$ for all $x, y \in \mathrm{qf}(R)$, which is the third defining condition of a valuation map. Thus, $v$ is a valuation map. The last part of statement (b) can be readily deduced from the fact that $G_+ = \{xR^\times : x \in R \setminus \{0\}\}$.

(b) $\Rightarrow$ (a): Suppose that $v\colon \mathrm{qf}(R) \to G \cup \{\infty\}$ is a valuation map satisfying that $R = \{x \in \mathrm{qf}(R) : v(x) \ge 0\}$. Take $x, y \in R$ to be nonzero elements. Then either $v(xy^{-1}) \in G_+$ or $v(yx^{-1}) \in G_+$, which implies that either $xy^{-1} \in R$ or $yx^{-1} \in R$. Hence $y \mid_R x$ or $x \mid_R y$. As a consequence, $R$ is a valuation domain.

For the final statement, one can easily check that $M := \{x \in R : v(x) > 0\}$ is an ideal of $R$. If a nonzero $u \in R$ has a positive valuation, then $v(u^{-1}) = -v(u) < 0$, and so $u \notin R^\times$. In addition, if $v(u) = 0$, then $v(u^{-1}) = 0$, which means that $u \in R^\times$. Hence $R^\times = R \setminus M$, which means that $M$ is a maximal ideal. $\square$

With notation as in the proof of Theorem 8, the group $G$ is called the *group of divisibility* of the integral domain $R$.

**Primes and Primary Ideals.** We proceed to discuss prime and primary ideals in the special setting of valuation domains.

**Proposition 9.** *In a valuation domain, every proper radical ideal is prime.*

*Proof.* Let $R$ be a valuation domain, and let $I$ be a proper radical ideal of $R$. Then $I = \operatorname{Rad} I$ is the intersection of all minimal prime ideals over $I$. Since the ideals of $R$ form a chain, there is only one minimal prime ideal $P$ over $I$. Hence we find, *a posteriori*, that $I = P$, and so $I$ is prime. $\square$

**Proposition 10.** *Let $R$ be a valuation domain, and let $I$ be a proper ideal of $R$. Then $\bigcap_{n\in\mathbb{N}} I^n$ is a prime ideal of $R$ that contains each prime ideal properly contained in $I$.*

*Proof.* Set $J := \bigcap_{n\in\mathbb{N}} I^n$. Take $x, y \in R$ such that $x \notin J$ and $y \notin J$, and then take $n \in \mathbb{N}$ such that $x, y \notin I^n$. Since $R$ is a valuation domain, $I^n \subseteq (x)$ and $I^n \subseteq (y)$. The former inclusion ensures that $I^n(y) \subseteq (xy)$. On the other hand, $I^n(y) \neq (xy)$ because $x \notin I^n$. Then $I^{2n} \subseteq I^n(y) \subsetneq (xy)$. This implies that $xy \notin I^{2n}$, and so $xy \notin J$. Hence $J$ is prime.

To argue the last statement, suppose that $P$ is a prime ideal properly contained in $I$. Since $P$ is prime, $I^n \not\subseteq P$ for any $n \in \mathbb{N}$. As $R$ is a valuation domain, $P \subseteq I^n$ for every $n \in \mathbb{N}$, which means that $P \subseteq J$. $\square$

Before discussing the configuration of primary ideals inside a valuation domain, we need the following lemma.

**Lemma 11.** *Let $R$ be a valuation domain, and let $I$ be a proper ideal of $R$. If $J$ is an ideal of $R$ such that $I \subsetneq \operatorname{Rad} J$, then $J$ contains a power of $I$.*

*Proof.* Exercise. $\square$

Recall that an ideal $I$ of $R$ is idempotent if $I^2 = I$. We conclude this lecture with the following result about primary ideals of a valuation domain.

**Theorem 12.** *Let $R$ be a valuation domain, and let $P$ be a prime ideal of $R$. Then the following statements hold.*

(1) *The product of $P$-primary ideals is a $P$-primary ideal.*
(2) *If $P$ is not idempotent, then the $P$-primary ideals of $R$ are the powers of $P$.*
(3) *The intersection of all $P$-primary ideals of $R$ is a prime ideal that contains each prime ideal properly contained in $P$.*

*Proof.* (1) Let $Q_1$ and $Q_2$ be $P$-primary ideals of $R$. Then $\operatorname{Rad}(Q_1Q_2) = (\operatorname{Rad} Q_1) \cap (\operatorname{Rad} Q_2) = P$. To argue that $Q_1Q_2$ is a primary ideal, take $x, y \in R$ such that $xy \in Q_1Q_2$ but $x \notin \operatorname{Rad}(Q_1Q_2) = P$. We claim that $(x)Q_1 = Q_1$. Since $x \notin Q_1$, the fact that $R$ is a valuation domain implies that $Q_1 \subsetneq (x)$. Set $J := (Q_1 : (x))$. Observe that the inclusion $Q_1 \subseteq (x)$ implies that $J \subseteq ((x) : (x)) = R$, and so $J$ is an ideal of $R$. Observe, in addition, that $J = x^{-1}Q_1$, that is, $Q_1 = (x)J$. Now the fact that $Q_1$ is primary with $x \notin P = \operatorname{Rad} Q_1$ ensures that $J \subseteq Q_1$. Therefore $Q_1 = J$, and so $(x)Q_1 = (x)J = Q_1$, as claimed. Then we see that $xy \in Q_1Q_2 = (x)Q_1Q_2$, which implies that $y \in Q_1Q_2$. Hence $Q_1Q_2$ is primary.

(2) Suppose that $P$ is not idempotent, that is, $P^2 \subsetneq P$. By part (1), the powers of $P$ are $P$-primary ideals. On the other hand, let $Q$ be a $P$-primary ideal. Since $P^2$ is contained in $\operatorname{Rad} Q$, Lemma 11 guarantees that $Q$ contains a power of $P^2$, and so we can take $n \in \mathbb{N}$ such that $P^n \subseteq Q$ but $P^{n-1} \not\subseteq Q$. Take $x \in P^{n-1}$ such that $x \notin Q$. The inclusion $Q \subsetneq (x)$ holds because $R$ is a valuation domain. Set $J := (Q : (x))$, and observe that $J$ is an ideal of $R$ and $Q = (x)J$ (see previous paragraph). Since $Q$ is $P$-primary and $x \notin Q$, the inclusion $J \subseteq P$ holds, whence $Q = (x)J \subseteq P^{n-1}P = P^n$. Thus, each $P$-primary ideal of $R$ is a power of $P$.

(3) Assume that $P$ is not the only $P$-primary ideal of $R$. Let $I$ be the intersection of all $P$-primary ideals, and let $Q$ be a $P$-primary ideal different from $P$. By part (1), the ideal $Q^n$ is $P$-primary for every $n \in \mathbb{N}$ and, therefore, $I \subseteq \bigcap_{n\in\mathbb{N}} Q^n$. Observe that, by Lemma 11, every $P$-primary ideal contains a power of $Q$, and so $I = \bigcap_{n\in\mathbb{N}} Q^n$. It follows now by Proposition 10 that $I$ is a prime ideal of $R$ that contains each prime ideal properly contained in $Q$. Finally, suppose that $P'$ is a prime ideal of $R$ such that $P' \subsetneq P$. Then for each $n \in \mathbb{N}$, the fact that $Q^n$ is $P$-primary guarantees that $Q^n \not\subseteq P'$ and, as $R$ is a valuation domain, $P' \subseteq Q^n$. Thus, $P' \subseteq \bigcap_{n\in\mathbb{N}} Q^n = I$. □

## Exercises

**Exercise 1.** *Let $R$ be a valuation domain, and let $I$ be a proper ideal of $R$. If $J$ is an ideal of $R$ such that $I \subsetneq \operatorname{Rad} J$, then $I^n \subseteq J$ for some $n \in \mathbb{N}$.*

**Exercise 2.** *Let $R$ be a valuation domain, and let $P$ be a nonzero prime ideal of $R$. Show that if $R$ contains a finitely generated $P$-primary ideal, then $P$ is the maximal ideal of $R$.*

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