Localization

Localization of Rings. Let $R$ be a commutative ring with identity. A multiplicative subset of $R$ is a submonoid of $(R \setminus \{0\}, \cdot)$. Let $S$ be a multiplicative subset of $R$. One can define the following relation on $R \times S$: $(r_1, s_1) \sim (r_2, s_2)$ for $(r_1, s_1), (r_2, s_2) \in R \times S$ provided that $(r_1s_2 - r_2s_1)s = 0$ for some $s \in S$. It is not hard to check that $\sim$ is indeed an equivalence relation on $R \times S$. We let $S^{-1}R$ denote the set of equivalence classes of $\sim$ and, for $r \in R$ and $s \in S$, we let $r/s$ denote the equivalence class of $(r, s)$.

Motivated by the standard addition and multiplication of rational numbers, we can now define for $r_1/s_1$ and $r_2/s_2$ in $S^{-1}R$ the following operations:

$$
\frac{r_1}{s_1} + \frac{r_2}{s_2} := \frac{r_1s_2 + r_2s_1}{s_1s_2} \quad \text{and} \quad \frac{r_1}{s_1} \cdot \frac{r_2}{s_2} := \frac{r_1r_2}{s_1s_2}.
$$

It is routine to verify that both operations are well defined and that $(S^{-1}R, +, \cdot)$ is a commutative ring with identity $1/1$.

Proposition 1. $(S^{-1}R, +, \cdot)$ is a commutative ring with identity.

The ring $S^{-1}R$ is called the localization of $R$ at $S$. We can easily see that the map $\pi: R \to S^{-1}R$ defined by $\pi(r) = r/1$ satisfies the properties in the following proposition.

Proposition 2. Let $R$ be a commutative ring with identity, and let $S$ be a multiplicative subset of $R$. Then the following statements hold.

1. The map $\pi: R \to S^{-1}R$ is a ring homomorphism satisfying that $\pi(s)$ is a unit in $S^{-1}R$ for every $s \in S$. In addition, $\pi$ is injective if and only if $S$ contains no zero-divisors of $R$.

2. If $\varphi: R \to T$ is a ring homomorphism such that $\varphi(s)$ is a unit in $T$ for every $s \in S$, then there exists a unique ring homomorphism $\theta: S^{-1}R \to T$ such that $\varphi = \theta \circ \pi$.

Proof. (1) One can readily see that $\pi$ is a ring homomorphism. For every $s \in S$, it is clear that $1/s \in S^{-1}R$ and, therefore, $\pi(s) = s/1$ is a unit in $S^{-1}R$. If $s \in S$ is a zero-divisor in $R$, then taking $r \in R \setminus \{0\}$ with $sr = 0$, we can see that $\pi(r) = 0$ and so $\pi$ is not injective. Conversely, if $\pi(r) = 0$ for some $r \in R \setminus \{0\}$, then $r/1 = 0/1$ and so there is an $s \in S$ such that $sr = 0$. 


(2) For \( \varphi \) as in (2), define \( \theta : S^{-1}R \rightarrow T \) by \( \theta(r/s) = \varphi(r)\varphi(s)^{-1} \). Since \( \varphi(s) \in T^\times \) for every \( s \in S \), the element \( \varphi(r)\varphi(s)^{-1} \) belongs to \( T \), and it is easy to check that \( \theta \) is a well-defined ring homomorphism. Since \( \theta(\pi(r)) = \theta(r/1) = \varphi(r) \), the equality \( \theta \circ \pi = \varphi \) holds. Finally, for any ring homomorphism \( \theta' : S^{-1}R \rightarrow T \) with \( \varphi = \theta' \circ \pi \), we see that \( \theta'(r/s) = \theta'(r/1)\theta'(1/s) = \theta'(\pi(r))\theta'(\pi(s))^{-1} = \varphi(r)\varphi(s)^{-1} = \theta(r/s) \) for all \( r/s \in S^{-1}R \). Hence \( \theta' = \theta \), and the uniqueness follows. \( \square \)

If \( R \) is an integral domain, then we can take \( S \) to be \( (R \setminus \{0\}, \cdot) \), then the localization of \( R \) at \( S \) is clearly a field. In this case, \( S^{-1}R \) is called the \textit{quotient field} of \( R \) and is denoted by \( qf(R) \). Note that \( \mathbb{Q} \) is the quotient field of \( \mathbb{Z} \). The following two examples of localizations show often in commutative ring theory.

**Example 3.** Let \( R \) be a commutative ring with identity, and let \( P \) be a prime ideal of \( R \). Since \( R \) is prime, \( S := R \setminus P \) is a multiplicative subset of \( R \). The ring \( S^{-1}R \) is called the \textit{localization of \( R \) at \( P \)} and is denoted by \( R_P \).

1. For instance, if \( p \in \mathbb{P} \), then \( \mathbb{Z}_{(p)} = \{ m/n : m, n \in \mathbb{Z} \text{ and } p \nmid n \} \);
   observe that the units of \( \mathbb{Z}_{(p)} \) are the elements \( m/n \) such that \( m, n \in \mathbb{Z} \) and \( p \nmid mn \).

2. Set \( R = \mathbb{C}[x, y] \) and \( P = (x, y) \). Then \( P \) is a prime ideal, and the localization \( R_P \) of \( R \) at \( P \) consists of all rational expressions \( f/g \), where \( f, g \in R \) and \( g \notin P \), that is, \( g(0, 0) \neq 0 \). The units of \( R_P \) are the rational expressions \( f/g \) satisfying \( f(0, 0)g(0, 0) \neq 0 \).

In general, the units of \( R_P \) have the form \( r/s \) with \( r, s \in R \) such that \( rs \notin P \).

**Example 4.** Let \( R \) be a commutative ring with identity, and let \( f \) be an element of \( R \) such that \( f^n \neq 0 \) for any \( n \in \mathbb{N}_0 \). For \( S := \{ f^n : n \in \mathbb{N}_0 \} \), the ring \( S^{-1}R = R[1/f] \) is often denoted by \( R_f \). It is not hard to argue that \( R_f \) is isomorphic to the ring \( R[x]/(xf - 1) \). For instance, \( \mathbb{Z}[x]_x = \mathbb{Z}[x, 1/x] \), which is the ring of Laurent polynomials in one variable over \( \mathbb{Z} \).

An integral domain is the intersection of all its localizations at prime ideals.

**Proposition 5.** If \( R \) is an integral domain, then \( R = \bigcap_P R_P = \bigcap_M R_M \), where the first intersection runs over all prime ideals of \( R \) and the second intersection runs over all maximal ideals of \( R \).

**Proof.** It is clear that \( R \subseteq \bigcap_P R_P \subseteq \bigcap_M R_M \). To show that \( \bigcap_M R_M \subseteq R \), take \( a \in \bigcap_M R_M \) and suppose, by way of contradiction, that \( a \notin R \). The set \( I_a := \{ r \in R : ra \in R \} \) is an ideal of \( R \), which is a proper ideal because \( a \notin R \). Let \( M \) be a maximal ideal of \( R \) containing \( I_a \). Then \( a \in R_M \), and we can take \( r \in R \) and \( s \in R \setminus M \) such that \( a = r/s \). As \( sa = r \in R \), we see that \( s \in I_a \subseteq M \), which is a contradiction. \( \square \)
Localization and Ideals. For an ideal $I$ of $R$, the ideal $S^{-1}R \pi(I)$ of $S^{-1}R$ is called the extension of $I$ by $\pi$ and is denoted by $S^{-1}I$. Observe that every element of $S^{-1}I$ can be written as $a/s$ for some $a \in I$ and $s \in S$.

**Proposition 6.** Let $R$ be a commutative ring with identity, and let $S$ be a multiplicative subset of $R$. Then the following statements hold.

1. For any ideal $J$ of $S^{-1}R$ the equality $S^{-1}\pi(J) = J$ holds. In particular, every ideal of $S^{-1}R$ is the extension of an ideal in $R$.
2. For an ideal $I$ of $R$, the equality $S^{-1}I = S^{-1}R$ holds if and only if $I \cap S \neq \emptyset$.
3. The assignment $I \mapsto S^{-1}I$ induces a bijection between the set of prime ideals of $R$ disjoint from $S$ and the set of prime ideals of $S^{-1}R$.

**Proof.** (1) It suffices to show that $J$ is contained in the ideal $J' := S^{-1}\pi^{-1}(J)$. Take $r/s \in J$. As $r/1 = (s/1)(r/s) \in J$, it follows that $r \in \pi^{-1}(J)$, and so $r/1 \in S^{-1}\pi^{-1}(J)$. Since $J'$ is an ideal of $S^{-1}R$, we see that $r/s = (1/s)(r/1) \in J'$. Hence $J' = J$. The second statement is an immediate consequence of the first one.

(2) If $S^{-1}I = S^{-1}R$, then $a/s = 1/1$ for some $a \in I$ and $s \in S$. So we can take $s' \in S$ such that $(a - s)s' = 0$. This means that $ss' = as' \in I$, whence $I \cap S = \emptyset$. Conversely, assume that $I \cap S \neq \emptyset$ and take $a \in I \cap S$. Then for all $r/s \in S^{-1}R$, we see that $r/a \in I$ while $sa \in S$, which implies that $r/s = (ra)/(sa) \in S^{-1}I$. Thus, $S^{-1}I = S^{-1}R$.

(3) Let $\mathcal{I}$ be the set of prime ideals in $R$ that are disjoint from $S$, and let $\mathcal{J}$ be the set of prime ideals in $S^{-1}R$. Let $e: \mathcal{I} \to \mathcal{J}$ and $c: \mathcal{J} \to \mathcal{I}$ be the maps given by the assignments $I \mapsto S^{-1}I$ and $J \mapsto \pi^{-1}(J)$, respectively. Since homomorphic inverse images of prime ideals are prime ideals, $c$ is well defined. To check that $e$ is also well defined, take $P \in \mathcal{I}$ and let us verify that $S^{-1}P$ is a prime ideal. Take $r_1, r_2 \in R$ and $s_1, s_2 \in S$ such that $(r_1/s_1)(r_2/s_2) \in S^{-1}P$. Then there are elements $a \in P$ and $s' \in S$ such that $(r_1 r_2 s - a s_1 s_2) s' = 0$, which implies that $r_1 r_2 s s' \in P$. As $P$ is prime and disjoint from $S$, we obtain that either $r_1 \in P$ or $r_2 \in P$, from which we deduce that either $r_1/s_1 \in S^{-1}P$ or $r_2/s_2 \in S^{-1}P$. Hence $S^{-1}P$ is a prime ideal, and so the map $e$ is well defined. Part (1) guarantees that $e \circ e$ is the identity of $\mathcal{J}$. Proving that $c \circ e$ is the identity of $\mathcal{I}$ amounts to arguing that $c(e(P)) \subseteq P$ for every $P \in \mathcal{I}$. To do so, take $a_3/s_3 \in e(P) = S^{-1}P$ for $a_3 \in P$ and $s_3 \in S$. If $r \in \pi^{-1}(a_3/s_3)$, then $r/1 = a_3/s_3$ and there is an $s'' \in S$ with $(rs_3 - a_3)s'' = 0$. This implies that $rs_3 \in P$, from which we deduce that $r \in P$. Hence $c(e(P)) \subseteq P$, as desired. Thus, $c \circ e$ is the identity of $\mathcal{I}$, which completes the proof.

The property of being Noetherian is preserved under localization.

**Proposition 7.** Let $R$ be a Noetherian domain, and let $S$ be a multiplicative subset of $R$. Then $S^{-1}R$ is also Noetherian.
Proof. By Proposition 6, any ideal of $S^{-1}R$ has the form $S^{-1}I$ for some ideal $I$ of $R$. Since $R$ is Noetherian, $I = Ra_1 + \cdots + Ra_n$ for some $a_1, \ldots, a_n \in R$. Then for each $a/s \in S^{-1}I$ with $a \in I$ and $s \in S$, we can write $a = \sum_{i=1}^n r_ia_i$ for some $r_1, \ldots, r_n \in R$ to obtain the equality $a/s = \sum_{i=1}^n (r_i/s)(a_i/1)$. Thus, $S^{-1}I$ is the ideal of $S^{-1}R$ generated by $a_1/1, \ldots, a_n/1$. Hence $S^{-1}R$ is a Noetherian ring. \hfill \Box

In addition, localization preserves the most important ideal operations, as we will see in the following proposition.

**Proposition 8.** Let $R$ be a commutative ring with identity, and let $S$ be a multiplicative subset of $R$. For ideals $I$ and $J$ of $R$, the following statements hold.

1. $S^{-1}(I + J) = S^{-1}I + S^{-1}J$.
2. $S^{-1}(I \cap J) = S^{-1}I \cap S^{-1}J$.
3. $S^{-1}(IJ) = (S^{-1}I)(S^{-1}J)$.
4. $S^{-1}R / S^{-1}I \cong S^{-1}(R/I)$.

*Proof.* Exercise. \hfill \Box

**Localization of Modules.** We can localize modules in the same way we have localized rings. Let $R$ be a commutative ring with identity with a multiplicative subset $S$, and let $M$ be an $R$-module. It is easy to verify that the relation on $M \times S$ defined by $(m_1, s_1) \sim (m_2, s_2)$ if there is an $s \in S$ such that $(m_1s_2 - m_2s_1)s = 0$ is an equivalence relation, and one denotes the class of $(m, s)$ by $m/s$ and the set of all equivalence classes by $S^{-1}M$. It is routine to verify that the operations

$$
\frac{m_1}{s_1} + \frac{m_2}{s_2} := \frac{s_2m_1 + s_1m_2}{s_1s_2} \quad \text{and} \quad \frac{r}{s} \cdot \frac{m_1}{s_1} := \frac{rm_1}{ss_1},
$$

where $m_1/s_1, m_2/s_2 \in S^{-1}M$ and $r/s \in S^{-1}R$, are well defined and turn $S^{-1}M$ into an $S^{-1}R$-module, which is called the *localization* $M$ at $S$. In particular, $S^{-1}M$ is an $R$-module. As Exercise 7 indicates, localization commutes with (direct) sums, intersections, and quotients of modules. The map $\pi: M \to S^{-1}M$ defined by $m \mapsto m/1$ is an $R$-module homomorphism and has the universal property described in Proposition 9(2).

**Proposition 9.** Let $R$ be a commutative ring with identity, let $S$ be a multiplicative subset of $R$, and let $M$ be an $R$-module. Then the following statements hold.

1. The map $\pi: M \to S^{-1}M$ defined by $m \mapsto m/1$ is an $R$-module homomorphism and $\ker \pi = \{m \in M : sm = 0$ for some $s \in S\}$.
2. If $M'$ is an $R$-module such that, for each $s \in S$, left multiplication by $s$ yields a bijection on $M'$ and, in addition, $\varphi: M \to M'$ is an $R$-module homomorphism, then there is a unique $R$-module homomorphism $\theta: S^{-1}M \to M'$ such that $\varphi = \theta \circ \pi$. 
Any $R$-module homomorphism $\psi: M \to M'$ induces an $S^{-1}R$-module homomorphism $S^{-1}M \to S^{-1}M'$ via the assignment $m/s \mapsto \psi(m)/s$.

**Proof.** Exercise. 

The localization of a Noetherian $R$-module is Noetherian.

**Proposition 10.** Let $R$ be a commutative ring with identity, and let $S$ be a multiplicative subset of $R$. If $M$ is a Noetherian $R$-module, then $S^{-1}M$ is also a Noetherian $S^{-1}R$-module.

**Proof.** See the proof of Proposition 7.

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**Exercises**

**Exercise 1.** Let $R$ be a commutative ring with identity, and let $S$ be a multiplicative subset of $R$. The set $\tilde{S} := \{r \in R : \pi(r) \text{ is a unit of } S^{-1}R\}$ is called the saturation of $S$. Prove the following statements.

1. $\tilde{S} = \{r \in R : rt \in S \text{ for some } t \in R\}$.
2. $\tilde{S}$ is a multiplicative subset of $R$ satisfying $S \subseteq \tilde{S} = \overline{\tilde{S}}$.
3. $S^{-1}R \cong \tilde{S}^{-1}R$.

**Exercise 2.** Let $R$ be a commutative ring with identity, and let $I$ and $J$ be ideals of $R$. Prove that $I = J$ if and only if $IR_P = JR_P$ for every maximal ideal $P$ of $R$.

**Exercise 3.** Let $R$ be an integral domain, and let $S$ be a multiplicative subset of $R$. Prove the following statements.

1. If $R$ is a UFD, then $S^{-1}R$ is a UFD.
2. Suppose that $S$ is saturated and $R$ is atomic (i.e., every nonzero nonunit of $R$ factors into irreducibles). If $S^{-1}R$ is a UFD, then $R$ is a UFD.

**Exercise 4.** Prove Proposition 8.

**Exercise 5.** Prove Proposition 9.

**Exercise 6.** Let $R$ be a commutative ring, and let $S$ be a multiplicative subset of $R$. Let $M$ be an $R$-module. Let $\pi: M \to S^{-1}M$ be the natural map. Prove the following statements.

1. For each $R$-submodule $N$ of $M$, the set $S^{-1}N := \{n/s : n \in N \text{ and } s \in S\}$ is an $S^{-1}R$-submodule of $S^{-1}M$.
2. If $L$ is an $S^{-1}R$-submodule of $S^{-1}M$, then $\pi^{-1}(L)$ is an $R$-submodule of $M$.
3. If $N$ is an $R$-submodule of $M$, then $N \subseteq \pi^{-1}(S^{-1}N)$. Also, if $N = \pi^{-1}(L)$ for an $S^{-1}R$-submodule $L$ of $S^{-1}M$, then $L = S^{-1}N$. In particular, every $S^{-1}R$-submodule of $S^{-1}M$ has the form $S^{-1}N$ for an $R$-submodule $N$ of $M$. 
(4) Deduce that there is a bijection between the set of $S^{-1}R$-submodules of $S^{-1}M$ and the set of $R$-submodules $N$ of $M$ satisfying the condition: if $sm \in N$ for some $s \in S$ and $m \in M$, then $m \in N$.

Exercise 7. Let $R$ be a commutative ring with identity, let $S$ be a multiplicative subset of $R$, and let $M$ be an $R$-module. For any submodules $M_1$ and $M_2$ of $M$, prove the following statements.

1. $S^{-1}(M_1 + M_2) = S^{-1}M_1 + S^{-1}M_2$.
2. $S^{-1}(M_1 \oplus M_2) = S^{-1}M_1 \oplus S^{-1}M_2$.
3. $S^{-1}(M_1 \cap M_2) \cong S^{-1}M_1 \cap S^{-1}M_2$.
4. $S^{-1}M / S^{-1}M_1 = S^{-1}(M/M_1)$.