MIT 18.211: COMBINATORIAL ANALYSIS

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Lecture 9: Integer Partitions II

We have previously established a recursive formula for the number of partitions of a set of a given size into a given number of parts (that is, for the Stirling numbers of the second kind). Let us establish a somehow parallel formula for number partitions.

Proposition 1. *For every $n \in \mathbb{N}$ and $k \in [n]$, the following recurrence identity holds:*

$$p_k(n) = p_{k-1}(n-1) + p_k(n-k). \tag{0.1}$$

Proof. By definition, the left-hand side of (0.1) counts the set P of partitions of n into k parts. To count the same set in a different way, write $P = P_1 \cup P_2$, where P_1 is the set of partitions of n into k parts whose last part is 1 and P_2 is the set of partitions of n into k parts whose last part is at least 2. Notice that P_1 is in bijection with the set of all partitions of $n-1$ into $k-1$ parts: this bijection consists in dropping the last part of the partition. Hence $|P_1| = p_{k-1}(n-1)$. On the other hand, P_2 is in bijection with the set of all partitions of $n-k$ into k parts: this time the bijection consists in dropping the first column of the corresponding Ferrer diagram. As a result, $|P_2| = p_k(n-k)$. Therefore (0.1) follows from the fact that $p_k(n) = |P_1| + |P_2|$. □

Now let $p(j, k, n)$ be the number of partitions of n into at most k parts whose largest part is at most j. The next proposition shows a connection between partitions and the q-analogs of the binomial coefficients. First, let us recall some aspects from linear algebra. A $k \times m$ matrix with entries in a field is in row-reduced echelon form if the following statements hold:

(1) the first nonzero entry of each row is 1,
(2) for each $i \in [k-1]$, the column of the first 1 of the $(i+1)$-th row is located to the right of the column of the first 1 of the i-th row,
(3) for each $i \in [k]$, the column containing the first 1 of the i-th row has only one nonzero entry.

As in the case of m-dimensional vector spaces over the field $\mathbb{R}$, any k-dimensional subspace of F^m, where F is a field, has a unique ordered basis $v_1, \dots, v_k$ such that the $k \times m$ matrix $M := (v_1, \dots, v_k)^T$ (where the superscript T is the transpose operator) is in row-reduced echelon form. We are in a position now to establish the following identity.

Proposition 2. *For all $j, k \in \mathbb{N}$, the following identity holds:*

$$\sum_{n \ge 0} p(j,k,n) q^n = \binom{j+k}{j}_q. \tag{0.2}$$

Proof. Set $m := j + k$, and let $\mathbb{F}_q$ be the field of q elements. Since the right-hand side of (0.2) counts the set of k-dimensional subspaces of the vector space $\mathbb{F}_q^m$, it suffices to show that the left-hand side counts the set of $k \times m$ matrices with entries in $\mathbb{F}_q$ that are in row-reduced echelon form. To do so, consider $a_1, \dots, a_k \in \mathbb{N}$ with $1 \le a_1 < a_2 < \dots < a_k \le m$, and let M be a row-reduced echelon form $k \times m$ matrix with entries in $\mathbb{F}_q$ such that, for each $i \in [k]$, the first 1 in the i-th row of M is in column a_i. For example, if $(a_1, a_2, a_3, a_4) = (2, 4, 5, 7)$, then

$$M = \begin{pmatrix} 0 & 1 & * & 0 & 0 & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{pmatrix},$$

where the asterisks inside the matrix M are arbitrary scalars in $\mathbb{F}_q$. Now consider $(b_1, \dots, b_k) \in \mathbb{N}_0^k$, where b_i is the number of asterisks in the i-th row of M. Observe that $b_i = m - a_i - (k - i) = j - (a_i - i)$ for every $i \in [k]$. Therefore $b_1 \ge b_2 \ge \dots \ge b_k$, and so $(b_1, \dots, b_k)$ determines a partition of some $n := b_1 + \dots + b_k \in \mathbb{N}_0$, which has at most k parts and whose largest part is at most $j = m - k$. Since each asterisk can take any of the q values of $\mathbb{F}_q$, the number of row-reduced echelon form matrices corresponding to the prescribed sequence $1 \le a_1 < a_2 < \dots < a_k \le m$ is q^n. On the other hand, for every partition $(b_1, \dots, b_k)$ of n into at most k parts with largest part at most j, we can set $a_i = j - b_i + i$ and there are q^n row-reduced echelon form matrices with e_i in the column a_i for every $i \in [k]$. Hence

$$\binom{j+k}{j}_q = \sum_{p \in P_{j,k}} q^{|p|} = \sum_{n \in \mathbb{N}_0} p(j,k,n) q^n,$$

where $|p|$ is the size of the partition p and $P_{j,k}$ denotes the set of partitions of n into at most k parts with largest part at most j. $\square$

Practice Exercises

Exercise 1. [1, Exercise 5.14] *Show that the following inequality holds for every $n \in \mathbb{N}$:*

$$\sum_{j=1}^{n} p(j) < p(2n).$$

Exercise 2. *For all $n \in \mathbb{N}$ and $(a_1, \dots, a_k) \vdash n$, the identity*

$$\sum_{j=1}^{k} (j-1)a_j = \sum_{i=1}^{a_1} \binom{b_i}{2},$$

holds, where $(b_1, \dots, b_{a_1})$ is the conjugate partition of $(a_1, \dots, a_k)$.

Exercise 3. *Use Proposition 2 to verify that $\binom{5}{3}_q = 1 + q + 2q^2 + 2q^3 + 2q^4 + q^5 + q^6$.*

References

[1] M. Bóna: *A Walk Through Combinatorics: An Introduction to Enumeration and Graph Theory* (Fourth Edition), World Scientific, New Jersey, 2017.

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