MIT 18.211: COMBINATORIAL ANALYSIS

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Lecture 8: Integer Partitions I

Let $n, a_1, \dots, a_k$ be positive integers with $a_1 \geq \dots \geq a_k$ and $n = a_1 + \dots + a_k$. Then the k-tuple $(a_1, \dots, a_k)$ is called a *partition* of n into k parts. When $(a_1, \dots, a_k)$ is a partition of n, we often write $(a_1, \dots, a_k) \vdash n$. The number of partitions of n into k parts is denoted by $p_k(n)$, while the number of partitions of n (without restricting the number of parts) is denoted by $p(n)$. Note that $p_k(n) = 0$ if $k \notin [n]$. We also assume that $p_k(0) = 0$ for every $k \in \mathbb{N}$ and, by convention, we set $p_0(0) = 1$ and $p(0) = 1$. We can easily see that $p_1(n) = p_n(n) = 1$ for every $n \in \mathbb{N}$ and also that $p_{n-1}(n) = 1$ provided that $n \geq 2$.

Example 1. Let us find a formula for $p_2(n)$. To do so, let (a_1, a_2) be a partition of n into two parts. Observe that any value $a_2 \in [\lfloor n/2 \rfloor]$ is permissible and determines the partition (a_1, a_2). On the other hand, if $a_2 > \lfloor n/2 \rfloor$, then we would obtain

$$n = a_1 + a_2 \geq (\lfloor n/2 \rfloor + 2) + (\lfloor n/2 \rfloor + 1) = 2\lfloor n/2 \rfloor + 3 > n,$$

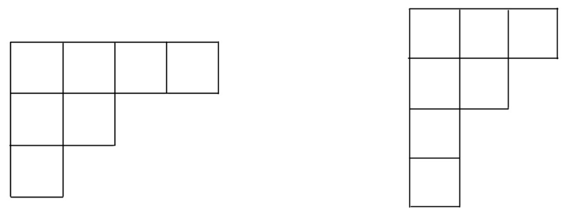
which is a contradiction. Hence we can conclude that $p_2(n) = \lfloor n/2 \rfloor$.

For $p = (a_1, \dots, a_k) \vdash n$, the *Ferrer* or *Young diagram* corresponding to p is a stack of k left-justified rows, where the i-th row consists of a_i adjacent unit squares. In this case, we call n the *size* of the Ferrer diagram. It is clear that there is a bijection between partitions of n and Ferrer diagrams (up to translation) of size n.

Example 2. The following diagrams are the Ferrer diagrams of $(4, 2, 1) \vdash 7$ and $(5, 3) \vdash 8$.

Observe that for $p \vdash n$, the reflection of the Ferrer diagram F of p with respect to the main diagonal line of F (i.e., the line with slope -1 passing through the top leftmost corner of F) is also a Ferrer diagram, and its corresponding partition is called the *conjugate partition* of p.

Example 3. The diagrams below are the Ferrer diagram of $(4, 2, 1) \vdash 7$ and the Ferrer diagram corresponding to the conjugate partition $(3, 2, 1, 1)$ of $(4, 2, 1)$.

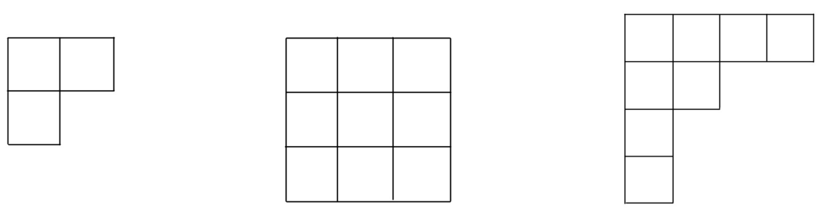


It follows from the definition of a conjugate partition that the size of the i-th row of (the Ferrer diagram of) a partition coincides with the size of the i-th column of (the Ferrer diagram of) its conjugate.

Proposition 4. *The number of partitions of n into (resp., at most) k parts equals the number of partitions of n whose first part is (resp., at most) k.*

Proof. As partitions of n are in bijection with Ferrer diagrams of size n, the statement of the proposition follows from the observation that a Ferrer diagram has (resp., at most) k rows if and only if its conjugate diagram has (resp., at most) k columns. □

A partition is called *self-conjugate* if it equals its conjugate, which happens precisely when the Ferrer diagram of the partition is symmetric with respect to its main diagonal line. The following diagrams are the Ferrer diagrams of the self-conjugate partitions $(2, 1) \vdash 3$, $(3, 3) \vdash 6$, and $(4, 2, 1, 1) \vdash 8$.



The *Durfee square* of a Ferrer diagram is the largest square anchored at the top leftmost corner of the diagram that is completely inside the diagram.

Proposition 5. *The number of partitions of n into distinct odd parts equals the number of self-conjugate partitions of n.*

Proof. Let S be the set of self-conjugate partitions of n, and let O be the set of partitions of n into distinct odd parts. Define $f \colon S \to O$ as follows. Take $p \in S$ with Ferrer diagram F and Durfee square D of side d. For each $i \in [\![1, d]\!]$, let H_i be the hook consisting of all unit squares of F that are located in either the i-th row or the i-th column of F. Observe that $|H_1| > |H_2| > \cdots > |H_d|$ and also that $|H_i|$ is odd for every $i \in [\![1, d]\!]$. Now define $f(p)$ to be the partition of n whose Ferrer diagram has d rows, the i-th row having size $|H_i|$. It is clear that $f \colon S \to O$ is an injective function.

Finally, note that f is also surjective as the method we described to assign a partition of O to any given partition of S can be easily reverted. Hence f is a bijection, and we can conclude that $|S| = |O|$. $\square$

Let $q(n)$ denote the number of partitions of n whose parts have size at least 2.

Proposition 6. *For each $n \in \mathbb{N}$ with $n \geq 2$, the following statements hold.*

(1) *$q(n)$ equals the number of partitions of n whose largest two parts are equal.*
(2) *$q(n) = p(n) - p(n-1)$.*

Proof. (1) This follows from the observation that the conjugate of a Ferrer diagram with every row consisting of at least two unit squares is a Ferrer diagram whose first two rows have the same number of unit squares.

(2) Let P_n and P_{n-1} denote the sets consisting of all Ferrer diagrams of size n and $n-1$, respectively. Define $f\colon P_{n-1} \to P_n$ by $f(a_1, \dots, a_k) = (a_1, \dots, a_k, 1)$ for every $(a_1, \dots, a_k) \vdash n-1$, that is, f is the map that adds a last part of size 1 to any given partition of $n-1$. It is clear that f is injective and, therefore, $|P_{n-1}| = |f(P_{n-1})|$. Observe that the partitions in $P_n \setminus f(P_{n-1})$ are precisely those partitions of n whose parts have size at least 2. Hence $|P_n \setminus f(P_{n-1})| = q(n)$. As a consequence,

$$p(n) = |P_n| = |P_n \setminus f(P_{n-1})| + |f(P_{n-1})| = q(n) + p(n-1),$$

and so we can conclude that $q(n) = p(n) - p(n-1)$. $\square$

Practice Exercises

Exercise 1. *For each $n \geq 4$, find a formula involving the $p(k)$'s for some of the $k \in [n-1]$ for the number of partitions of n whose parts have size at least 3.*

Exercise 2. *Find the number of partitions of n whose third part is 2.*

Exercise 3. *Prove that for every $n \in \mathbb{N}$ the inequality $p(n)^2 < p((n+1)^2)$ holds.*

Exercise 4. *For $k, n \in \mathbb{N}$, argue that $p_1(n) + p_2(n) + \cdots + p_k(n) = p_k(n+k)$.*

References

[1] M. Bóna: *A Walk Through Combinatorics: An Introduction to Enumeration and Graph Theory* (Fourth Edition), World Scientific, New Jersey, 2017.

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