MIT 18.211: COMBINATORIAL ANALYSIS

FELIX GOTTI

Lecture 7: Set Partitions

In this section we introduce set partitions and Stirling numbers of the second kind. Recall that two sets are called disjoint when their intersection is empty. A *partition* of a set S is a collection $\pi := \{B_1, \dots, B_k\}$ consisting of pairwise disjoint nonempty subsets of S such that $S = \bigcup_{j=1}^{k} B_j$. For each $j \in [\![1, k]\!]$, the set B_j is called a *block* of the partition π, and we write $|\pi| = k$ when π consists of k blocks. In addition, $S(n, k)$ denotes the number of partitions of $[n]$ having k blocks, and it is called a *Stirling number of the second kind* (we will define Stirling numbers of the first kind in coming lectures).

Let us take a look at some particular cases of the $S(n, k)$. By convention, we assume that $S(0, 0) = 1$. Observe that $S(n, k) = 0$ when $k > n$. In addition, there is only one partition of $[n]$ consisting of 1 block (as such a block must be the whole set $[n]$) and there is only one partition of $[n]$ consisting of n blocks (as each block is forced to have size one). Thus, $S(n, 1) = S(n, n) = 1$.

Example 1. We claim that $S(n, 2) = 2^{n-1} - 1$. Indeed, each partition of $[n]$ into two blocks, namely $\pi := \{B_1, B_2\}$, can be constructed by first choosing the subset B_1 in $2^n - 2$ ways (as B_1 cannot be neither empty nor the whose set $[n]$), which forces $B_2 = [n] \setminus B_1$, and then dividing our number of choices, $2^n - 2$, by 2 to account for the fact that the order of the blocks inside the partition π is irrelevant. Hence $S(n, 2) = 2^{n-1} - 1$.

Example 2. Let us verify now that $S(n, n-1) = \binom{n}{2}$. Indeed, every partition of $[n]$ into $n-1$ blocks must contain exactly one block of size 2, which completely determines the rest of the blocks, namely the remaining $n - 2$ blocks of size 1. Therefore the set of partitions of $[n]$ into two blocks is in bijection with the set of subsets of $[n]$ of size 2. Hence $S(n, n-1) = \binom{n}{2}$.

We can compute the number $S(n, k)$ recursively, as the following theorem indicates.

Theorem 3. *For any $n, k \in \mathbb{N}$ with $k \le n$, the following identity holds:*

$$S(n, k) = S(n-1, k-1) + kS(n-1, k). \tag{0.1}$$

Proof. By the definition of the Stirling numbers of the second kind, the left-hand side of (0.1) counts the set of partitions of $[n]$ into k blocks. We will count the same set by splitting it into two types of partitions: the partitions where n is itself a block and the partitions where the block containing n has size at least two. To count the partitions where n is a block by itself, we can take n out, choose a partition of $[n-1]$ into $k-1$ blocks in $S(n-1,k-1)$ ways, and enlarge the chosen partition to obtain a partition of $[n]$ into k blocks by adding $\{n\}$ as the n-th block. To count the partitions where the block containing n has size at least two, choose a partition of $[n-1]$ into k blocks in $S(n-1,k)$ different ways, and for each of such choices create a partition of $[n]$ into k blocks in k different ways by placing n inside one of the k blocks. Putting all together, we see that the number of partitions of $[n]$ into k blocks is $S(n-1,k-1)+kS(n-1,k)$, the right-hand side of (0.1). $\square$

We can express the number of surjective functions between finite sets in terms of the Stirling numbers of the second kind.

Proposition 4. *For every $n,k \in \mathbb{N}$, the number of surjective functions $f\colon [n] \to [k]$ is $S(n,k)k!$.*

Proof. To count the surjective functions $f\colon [n] \to [k]$, we can first fix a partition $\pi = \{B_1,\dots,B_k\}$ of $[n]$ into k blocks in $S(n,k)$ ways, then make a linear arrangement $w_1w_2\cdots w_k$ with the elements of $[k]$ in $k!$ ways, and then set $f^{-1}(w_i) = B_i$. Hence there are $S(n,k)k!$ surjective functions $f\colon [n] \to [k]$. $\square$

Stirling numbers of the second kind satisfy the following polynomial identity.

Proposition 5. *For every $n \in \mathbb{N}$, the following polynomial identity holds:*

$$x^n = \sum_{k=0}^{n} S(n,k)(x)_k, \tag{0.2}$$

where $(x)_k = x(x-1)\cdots(x-k+1)$.

Proof. First, assume that x belongs to $\mathbb{N}$. Observe that the left-hand side of (0.2) counts the functions $f\colon [n] \to [x]$. We can also count such functions as follows. For each $k \in [\![0,n]\!]$, we count the functions $f\colon [n] \to [x]$ with $|f([n])| = k$, which amounts to choosing a k-subset S of $[x]$ in $\binom{x}{k}$ different ways and then counting the set of surjective functions from $[n]$ to S, which we can verify that is $S(n,k)k!$ by mimicking the proof of Corollary 4. Hence there are $\sum_{k=0}^{n}\binom{x}{k}S(n,k)k! = \sum_{k=0}^{n} S(n,k)(x)_k$, and so (0.2) holds for every $n \in \mathbb{N}$. Therefore the polynomial $x^n - \sum_{k=0}^{n} S(n,k)(x)_k$ has degree at most n and more than n different roots, which implies that it must be the zero polynomial. Hence (0.2) must hold for every $x \in \mathbb{R}$. $\square$

For $n \in \mathbb{N}$, the total number of partitions of $[n]$ is denoted by $B(n)$ and called a *Bell number*. Then the equality $B(n) = \sum_{k=1}^{n} S(n,k)$ holds. We can compute Bell numbers recursively using the following recurrence identity.

Theorem 6. *For every $n \in \mathbb{N}_0$, the following identity holds:*

$$B(n+1) = \sum_{j=0}^{n} \binom{n}{j} B(j). \tag{0.3}$$

Proof. By the definition of a Bell number, the left-hand side of (0.3) counts the set of partitions of $[n+1]$. We can count the same set as follows. For each $s \in [\![1, n+1]\!]$, we can count the partitions of $[n+1]$ where the block B containing $\{n+1\}$ has size s: first choose in $\binom{n}{s-1}$ ways the elements in B that are different from $n+1$, then create a partition of $[n+1] \setminus B$ in $B(n+1-s)$ ways. Therefore

$$B(n+1) = \sum_{s=1}^{n+1} \binom{n}{s-1} B(n+1-s) = \sum_{j=0}^{n} \binom{n}{n-j} B(j) = \sum_{j=0}^{n} \binom{n}{j} B(j).$$

$\square$

Practice Exercises

Exercise 1. *For $n \in \mathbb{N}$ with $n \geq 3$, find a formula for $S(n,3)$.*

Exercise 2. *Prove that $S(n, n-2) = \binom{n}{3} + 3\binom{n}{4}$ for every $n \in \mathbb{N}$ with $n \geq 2$.*

Exercise 3. *Argue combinatorially that the number of partitions of $[n]$ with no two consecutive numbers in any block is $B(n-1)$.*

References

[1] M. Bóna: *A Walk Through Combinatorics: An Introduction to Enumeration and Graph Theory* (Fourth Edition), World Scientific, New Jersey, 2017.

Department of Mathematics, MIT, Cambridge, MA 02139
Email address: `fgotti@mit.edu`