MIT 18.211: COMBINATORIAL ANALYSIS

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Lecture 6: Compositions

Let us introduce the notion of a composition. A k-tuple $(a_1, \dots, a_k)$ in $\mathbb{N}_0^k$ that is a solution of the equation $x_1 + \dots + x_k = n$ is called a *weak composition* of n into k parts or a *weak k-composition* of n. If, in addition, each a_i is positive, then $(a_1, \dots, a_k)$ is called a *composition* of n into k parts or a *k-composition* of n.

Proposition 1. *For $n, k \in \mathbb{N}_0$, the following statements hold.*

1. *There are $\binom{n+k-1}{n}$ weak k-compositions of n.*
2. *There are $\binom{n-1}{k-1}$ k-compositions of n.*

Proof. (1) Note that we can interpret each weak k-composition of n as a way to place n identical balls (the n copies of 1 adding up to n) into k different boxes (the k coordinates of the weak k-composition). Therefore the number of weak k-compositions of n is $\binom{n+k-1}{n}$.

(2) Observe that a k-tuple $(a_1, \dots, a_k) \in \mathbb{N}^k$ is a k-composition of n if and only if $(a_1 - 1, \dots, a_k - 1) \in \mathbb{N}_0$ is a weak k-composition of $n - k$. Thus, it follows from part (1) that the number of k-compositions of n is $\binom{(n-k)+k-1}{n-k} = \binom{n-1}{k-1}$. $\square$

As a consequence, we can collect the following corollary.

Corollary 2. *For every $n \in \mathbb{N}$, there are 2^{n-1} compositions of n.*

Proof. Observe that a composition of n can have at most n parts. Thus, we can count the set of compositions of n by counting, for each $k \in [n]$, all compositions of $[n]$ into k parts. Then Proposition 1, along with the binomial theorem, ensures that

$$\sum_{k=1}^{n} \binom{n-1}{k-1} = \sum_{j=0}^{n-1} \binom{n-1}{j} = 2^{n-1},$$

whence there are 2^{n-1} compositions of n. $\square$

Observe that there is no point in counting the total number of weak compositions of $n \in \mathbb{N}$, as we can fill as many coordinates as we wish with zeros to obtain infinitely many weak compositions of n.

We can also interpret a weak k-composition of n as a linear arrangement of n identical balls and k identical bars, and this can also be used to give an alternative proof of part (1) of Proposition 1.

Example 3. For each weak k-composition $(a_1, \dots, a_k)$ of n, we can replace in the left-hand side of the identity $a_1 + \cdots + a_k = n$, each a_i by a_i consecutive stars and each $+$ by a bar, obtaining a linear arrangement of n identical stars and $k-1$ identical bars. Reversing this method, we can recover a weak k-composition of n out of a given linear arrangement of n identical stars and $k-1$ identical bars. Observe now that counting the linear arrangements of n stars and $k-1$ bars amounts to choosing the n positions to place the stars out of $n+k-1$ positions, which means that we can do this in $\binom{n+k-1}{n}$ different ways. As a result, we rediscovered that the number of weak k-compositions of n is $\binom{n+k-1}{n}$. The method shown in this paragraph is sometimes referred to as a "stars and bars" method.

Example 4. [1, Section 5.1] For $n \in \mathbb{N}$, suppose we want to compute the number of compositions of $n \in \mathbb{N}$ with last component different from 2. Well, if $n = 1$, then there is only one composition of n, and its last (first) component is 1, which is different from 2. If $n = 2$, then the compositions of 2 are (2) and $(1, 1)$, and only one of these compositions has its last component different from 2. Now assume that $n \geq 3$. We will count instead the compositions of n whose last components equal 2. Observe that $(a_1, \dots, a_k)$ is a composition of n whose last component is 2 (that is, $a_k = 2$) if and only if $(a_1, \dots, a_{k-1})$ is a composition of $n-2$. Therefore it follows from Corollary 2 that there are 2^{n-3} compositions of n whose last components equal 2. This allows us to conclude, using again Corollary 2, that the number of compositions of n whose last components differ from 2 is $2^{n-1} - 2^{n-3}$.

Example 5. Now suppose that we want to find the number of weak 5-compositions of 10 with exactly two parts being 0. We can first choose 2 out of the 5 parts in $\binom{5}{2}$ ways, and then we can fill the remaining positions with three positive integers adding up to 10 in $\binom{10-1}{3-1}$ ways, the number of 3-compositions of 10. Hence the desired number is $\binom{5}{2}\binom{9}{2}$.

Practice Exercises

Exercise 1. [1, Exercise 5.26] *A student is supposed to take* 18 *class hours per week (from Monday to Friday), and she wants to take fewer hours on Fridays than on Thursdays. In how many ways can she distribute the* 18 *hours through the five weekdays?*

Exercise 2. *Let k, n, r be positive integers such that $2k = n - r$. Prove that the number of compositions of n with r odd parts and s even parts is $\binom{r+s}{r}\binom{r+k-1}{r+s-1}$.*

Exercise 3. [1, Exercise 5.38] *Let b_n be the number of compositions of n with odd first part. Show that $b_n + b_{n+1} = 2^n$ for every $n \in \mathbb{N}$.*

References

[1] M. Bóna: *A Walk Through Combinatorics: An Introduction to Enumeration and Graph Theory* (Fourth Edition), World Scientific, New Jersey, 2017.

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