MIT 18.211: COMBINATORIAL ANALYSIS

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Lecture 4: Binomial and Multinomial Theorems

In this lecture, we discuss the binomial theorem and further identities involving the binomial coefficients. At the end, we introduce multinomial coefficients and generalize the binomial theorem.

Binomial Theorem. At this point, we all know beforehand what we obtain when we unfold $(x+y)^2$ and $(x+y)^3$. We can actually use binomial coefficients to generalize the formulas for the square and cube of a binomial expression.

Theorem 1. *For any $n \in \mathbb{N}_0$, the following identity holds:*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}. \tag{0.1}$$

Proof. Observe that each of the summands of the form $x^k y^{n-k}$ (for $k \in [\![0, n]\!]$) that we obtain after unfolding the left-hand side $(x+y)(x+y)\cdots(x+y)$ is the result of choosing either x or y in each of the n factors $x+y$ and multiplying out all the chosen variables. Then for a fixed k, the number of copies of the summand $x^k y^{n-k}$ we will obtain equals the number of ways of choosing k copies of x out of the n copies of $x+y$ (note that once we choose k copies of x, we are forced to choose $n-k$ copies of y). Thus, we will get $\binom{n}{k}$ copies of $x^k y^{n-k}$ for every $k \in [\![0, n]\!]$, and the identity (0.1) follows. □

Now we can evaluate the identity (0.1) to obtain interesting binomial identities. For instance,

- $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ results from taking $x = y = 1$ in (0.1), and
- $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$ results from taking $-x = y = 1$ in (0.1).

Example 2. A *lattice path* of length n is a sequence of lattice points $p_0, p_1, \dots, p_n$ in $\mathbb{N}_0^2$ such that $p_0 = (0,0)$ and $p_i - p_{i-1}$ is either $(1,0)$ or $(0,1)$ for every $i \in [\![1, n]\!]$. This can be pictured as a northeastern path from $p_0 = (0,0)$ to $p_n = (n-k, k)$ consisting of $n-k$ horizontal unit steps and k vertical unit steps. For instance, Figure 1 shows a lattice path of length 13 with 5 vertical unit steps. Observe that the set of lattice paths of length n is in bijection with the set of length-n binary sequences: label the

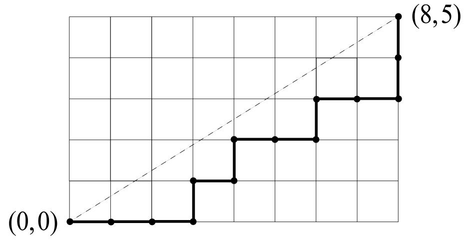

FIGURE 1. A 13-lattice path.

horizontal and vertical steps of a given lattice path by 0 and 1, respectively, and obtain the desired binary sequence by reading the path in northeastern direction. Conversely, given a binary sequence, we can recover its lattice path by sequentially interpreting the 0's and 1's as horizontal and vertical unit steps, respectively. Hence there are $\binom{n}{k}$ lattice paths of length n with k vertical steps. On the other hand, it is clear that there are 2^n lattice path of length n. Thus, counting lattice paths yields another way to argue the identity $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

Further Binomial Identities. We proceed to give further binomial identities that are often useful.

Proposition 3 (Vandermonde Identity)**.** *For any $m, n, k \in \mathbb{N}_0$, the following identity holds:*

$$\binom{m+n}{k} = \sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i}. \tag{0.2}$$

Proof. Suppose we have a group of $m+n$ students, where m are freshmen and n are sophomores. By definition, the left-hand side of (0.2) counts the number of k-teams we can form with the $m+n$ students. On the other hand, for each $i \in [\![0, k]\!]$, we can form $\binom{m}{i}\binom{n}{k-i}$ k-teams containing exactly i freshmen: we can first choose the i freshmen in $\binom{m}{i}$ ways and then complete the team choosing $k-i$ sophomores in $\binom{n}{k-i}$ ways. Hence the right-hand side of (0.2) also counts the number of k-teams we can form with the $m+n$ students. $\square$

Here are two more binomial identities.

Proposition 4. *For any $n, k \in \mathbb{N}_0$, the following identities hold:*

(1) $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$;
(2) $\binom{n+1}{k+1} = \sum_{j=k}^{n} \binom{j}{k}$.

Proof. (1) The left-hand side counts all the lattice paths ending at $(n-k, k+1)$. Now observe that the number of lattice paths ending at $(n-k, k+1)$ whose last step is vertical equals the number of lattice paths ending at $(n-k, k)$, and so there are $\binom{n}{k}$ of them. Similarly, the number of lattice paths ending at $(n-k, k+1)$ whose last step is horizontal equals the number of lattice paths ending at $(n-k-1, k+1)$, and so there are $\binom{n}{k+1}$ of them. Thus, the identity follows.

(2) The left-hand side counts the number of $(k+1)$-subsets of $[n+1]$. On the other hand, for each $i \in [\![k+1, n+1]\!]$, we can count the number of $(k+1)$-subsets of $[n+1]$ whose maximum is $i+1$ in $\binom{i}{k}$ different ways: first choose $i+1$ and then complete the $(k+1)$-subset by choosing k elements from $[i]$. Since every $(k+1)$-subset of $[n+1]$ has a maximum in the set $[\![k+1, n+1]\!]$, the right-hand side also counts the number of $(k+1)$-subsets of $[n+1]$. $\square$

Multinomial Theorem. Our next goal is to generalize the binomial theorem. First, let us generalize the binomial coefficients. For n identically-shaped given objects and k colors labeled by $1, 2, \dots, k$, suppose that there are a_i objects of color i for every $i \in [k]$. Then we let $\binom{n}{a_1, \dots, a_k}$ denote the number of ways of linearly arranging the n given objects.

Proposition 5. *Let $a_1, \dots, a_k$ be nonnegative integers, and set $n = a_1 + \dots + a_k$. Then*

$$\binom{n}{a_1, \dots, a_k} = \prod_{j=1}^{k} \binom{n - \sum_{i=1}^{j-1} a_i}{a_j} = \frac{n!}{a_1! a_2! \cdots a_k!}. \tag{0.3}$$

Proof. Suppose that we have n identically-shaped given objects of k colors with a_i of color i for each $i \in [k]$. We can linearly arrange these objects as follows: out of n given positions, choose a_1 in $\binom{n}{a_1}$ to place the objects of color 1, then out of the remaining $n - a_1$ positions choose a_2 in $\binom{n-a_1}{a_2}$ ways to place the objects of color 2, and so on, where the step j consists in choosing a_j positions in $\binom{n-(a_1+\dots+a_{j-1})}{a_j}$ ways from the remaining $n - (a_1 + \dots + a_{j-1})$ positions. Therefore the first equality in (0.3) holds. The second equality follows immediately after cancellation. $\square$

We conclude by stating the multinomial theorem. By virtue of the first equality in (0.3), the proof of the following theorem follows by mimicking that we gave for the binomial theorem, and so it is left to the reader as a practice exercise.

Theorem 6. *For any $n \in \mathbb{N}_0$, the following identity holds:*

$$(x_1 + \dots + x_k)^n = \sum_{a_1 + \dots + a_k = n} \binom{n}{a_1, \dots, a_k} x_1^{a_1} \cdots x_k^{a_k}.$$

Proof. Exercise. $\square$

Practice Exercises

Exercise 1. *Prove that* $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$ *for every* $n \in \mathbb{N}$.

Exercise 2. [1, Exercise 4.24] *Prove that the number of lattice paths from* $(0,0)$ *to* (n,n) *that never go above the line* $x = y$ *is* $\frac{1}{n+1}\binom{2n}{n}$.

Exercise 3. *Prove that for all* $k, n \in \mathbb{N}_0$, *the following identity holds:*

$$\binom{2n}{2k}\binom{2n-2k}{n-k}\binom{2k}{k} = \binom{2n}{n}\binom{n}{k}^2.$$

Exercise 4. [1, Exercise 4.34] *Argue combinatorially that* $\binom{3n}{n,n,n}$ *is divisible by* 6.

References

[1] M. Bóna: *A Walk Through Combinatorics: An Introduction to Enumeration and Graph Theory* (Fourth Edition), World Scientific, New Jersey, 2017.

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