MIT 18.211: COMBINATORIAL ANALYSIS

FELIX GOTTI

Lecture 38: Incidence Algebras and Möbius Inversion Formula

In this section we will introduce the incidence algebra of a (locally finite) poset, and we use this algebra to count the set of chains of the poset. We also use incidence algebras to prove the Möbius Inversion Formula.

Incidence Algebras. Let P be a locally finite poset, and let $I(P)$ denote the set consisting of all intervals of P. Now set

$$A(P) := \{\text{all functions } f \colon I(P) \to \mathbb{R}\}.$$

For $r, s \in P$ with $r \le s$, we write $f(r,s)$ instead of the more cumbersome notation $f([r,s])$. With the standard addition of functions and scalar product (that is, left-multiplication by a real number), $A(P)$ is a vector space over $\mathbb{R}$. Now we define a multiplication in $A(P)$ as follows:

$$(fg)(r,s) := \sum_{r \le x \le s} f(r,x)g(x,s)$$

for all $f, g \in A(P)$ and $r, s \in P$ with $r \le s$. This multiplication operation turns $A(P)$ into an algebra, that is, a vector space with a compatible multiplication (compatible means that multiplication is associative and distributes with addition). Now we let $\mathtt{1}$ denote the function $\mathtt{1} \colon I(P) \to \mathbb{R}$ given by $\mathtt{1}(r,s) = 1$ if $r = s$ and $\mathtt{1}(r,s) = 0$ if $r < s$. We can easily see that $\mathtt{1}f = f\mathtt{1} = f$ for every $f \in A(P)$, which means that $\mathtt{1}$ is the identity element of $A(P)$. We proceed to characterize the invertible elements of $A(P)$.

Proposition 1. *Let P be a locally finite poset. For $f \in A(P)$, the following conditions are equivalent.*

(1) *f has a right inverse.*

(2) *f has a left inverse.*

(3) *f is invertible.*

(4) *$f(r,r) \neq 0$ for all $r \in P$.*

Proof. Suppose first that f has a right inverse, namely, $g \in A(P)$. Then for every $r \in P$, the equality $1 = f(r,r)g(r,r)$ holds, which implies that $f(r,r) \neq 0$ and so, for every $s \in P$ with $r < s$, the equality $0 = \sum_{r \le x \le s} f(r,x)g(x,s)$ can be written as

$$g(r,s) = -f(r,r)^{-1} \sum_{r < x \le s} f(r,x)g(x,s). \tag{0.1}$$

In particular, (1) implies (4). On the other hand, if we assume that $f(r,r) \neq 0$ for all $r \in P$, then we can consider the element $g \in A(P)$ defined by $g(r,r) = f(r,r)^{-1}$ and by (0.1) for any $r, s \in P$ with $r < s$. It follows from the definition of g that g is a right inverse of f. Hence (4) implies (1). In a completely similar manner, we can show that (2) and (4) are equivalent statements. Finally, observe that if g_1 and g_2 are a left and right inverses of f, respectively, then $g_1 = g_1(fg_2) = (g_1 f)g_2 = g_2$. Thus, (3) is equivalent to the join statement of (1) and (2), which implies that (3) and (4) are equivalent statements. $\square$

It is not hard to verify that if $f_1, \dots, f_k \in A(P)$, then we can extend (by induction) the formula used to define a multiplication in $A(P)$ as follows:

$$(f_1 f_2 \dots f_k)(r,s) = \sum_{r = r_0 \le r_1 \le \dots \le r_k = s} f_1(r_0, r_1) f_2(r_1, r_2) \cdots f_k(r_{k-1}, r_k). \tag{0.2}$$

We leave this as an exercise (Exercise 1). It turns out that we can count both multichains and chains of any interval of P by using a very simple element of $A(P)$. Let $\eta \colon I(P) \to \mathbb{R}$ be the element of $A(P)$ defined by $\eta(r,s) = 1$ for any $r, s \in P$ with $r \le s$. Observe that

$$\eta^2(r,s) = \sum_{r \le x \le s} \eta(r,x)\eta(x,s) = \sum_{r \le x \le s} 1 = |[r,s]|.$$

In fact, for all $r, s \in P$ with $r \le s$ and $\ell \in \mathbb{N}$, it follows from the identity (0.2) that $\eta^\ell(r,s)$ counts the set of multichains of length ℓ in the interval $[r,s]$. Can we count chains of length ℓ in a somehow similar manner? The answer is yes.

Proposition 2. *Let P be a locally finite poset. For $r, s \in P$ with $r \le s$, the following statements hold.*

(1) $(\eta - 1)^\ell(r,s)$ *equals the number of chains from r to s having length ℓ.*

(2) $2 - \eta$ *is an invertible element of $A(P)$, and $(2-\eta)^{-1}(r,s)$ equals the number of chains from r to s.*

Proof. (1) Consider the element $\eta - 1 \in A(P)$ and observe that $(\eta - 1)(r,s) = 1$ if $r < s$ and $(\eta - 1)(r,s) = 0$ if $r = s$. Therefore

$$\begin{aligned}(\eta - 1)^\ell(r,s) &= \sum_{r=r_0\le r_1\le\cdots\le r_\ell=s} (\eta-1)(r_0,r_1)\cdots(\eta-1)(r_{\ell-1},r_\ell)\\ &= \sum_{r=r_0< r_1<\cdots< r_\ell=s} (\eta-1)(r_0,r_1)\cdots(\eta-1)(r_{\ell-1},r_\ell)\\ &= \sum_{r=r_0< r_1<\cdots< r_\ell=s} 1,\end{aligned}$$

which is precisely the number of chains from r to s having length ℓ (the second equality above follows from the fact that $(\eta - 1)(t,t) = 0$ for all $t \in P$).

(2) The first part of the statement follows from Proposition 1 as $(2 - \eta)(r,s) = 1$ if $r = s$ and $(2-\eta)(r,s) = -1$ if $r < s$. Fix $r, s \in P$ with $r \le s$, and suppose that the length of the longest chain in $[r,s]$ is ℓ. Now fix a subinterval $[x,y]$ of $[r,s]$. Then it follows form part (1) that $(\eta - 1)^{\ell+1}(x,y) = 0$, and so

$$(1 - (\eta-1)^{\ell+1})(x,y) = 1(x,y) + (\eta-1)^{\ell+1}(x,y) = 1(x,y).$$

Since $2 - \eta = 1 - (\eta - 1)$, we see that

$$(2-\eta)(1 + (\eta-1) + (\eta-1)^2 + \cdots + (\eta-1)^\ell)(x,y) = (1-(\eta-1)^{\ell+1})(x,y) = 1(x,y).$$

Therefore $(2-\eta)^{-1} = 1 + (\eta - 1) + \cdots + (\eta-1)^\ell$ in the incidence algebra $A([r,s])$, and this implies that $(2-\eta)^{-1}(r,s)$ equals the number of chains from r to s. □

Möbius Inversion Formula. Let P be a poset. A subset S of P is called an *(order) ideal* if for each $s \in S$, the fact that $r \le s$ in P implies that $r \in S$. For instance, the set $\Lambda_s := \{r \in P \mid r \le s\}$ is an ideal of P for every $s \in P$. Ideals of the form Λ_s for any $s \in P$ are called *principal (order) ideals*.

Now assume that P is a locally finite poset. It follows from Proposition 1 that $\eta \in A(P)$ is an invertible element. Let μ be the inverse of η in $A(P)$. This means that $\mu(r,r) = 1$ for all $r \in P$ and

$$\mu(r,s) = -\sum_{r\le x<s} \mu(r,x). \tag{0.3}$$

The function $\mu\colon I(P) \to \mathbb{R}$ is called the *Möbius function* of P. The Möbius function allows us to invert certain identity summations as follows.

Theorem 3 (Möbius Inversion Formula)**.** *Let P be a poset, where every principal order ideal is finite. For any $f, g\colon P \to \mathbb{R}$ and $s \in P$,*

$$f(s) = \sum_{r\le s} g(r) \quad \Leftrightarrow \quad g(s) = \sum_{r \le s} f(r)\mu(r,s).$$

Proof. For any $\nu \in A(P)$ and $h\colon P \to \mathbb{R}$, we can define $h\nu\colon P \to \mathbb{R}$ as follows:

$$(h\nu)(s) = \sum_{r \le s} h(r)\nu(r,s) \tag{0.4}$$

for all $s \in P$. One can readily check that for all $h\colon P \to \mathbb{R}$ and for all $\alpha, \beta \in A(P)$, both identities $h\mathbf{1} = h$ and $h(\alpha\beta) = (h\alpha)\beta$ hold. By virtue (0.4), it suffices to argue that $f = g\eta$ if and only if $g = f\mu$. This is indeed the case because the equality $f = g\eta$ holds if and only if

$$f\mu = (g\eta)\mu = g(\eta\mu) = g\mathbf{1} = g.$$

□

Example 4. The set $\mathbb{N}_0$ is clearly a locally finite poset with respect to the standard order. Let μ be the Möbius function of $\mathbb{N}_0$. Then $\mu(n,n) = 1$ and $\mu(n,n+1) = -\mu(n+1,n+1) = -1$ for every $n \in \mathbb{N}_0$. In addition, from the identity

$$\begin{aligned}\mu(n,n+k) &= -\mu(n+k,n+k) - \mu(n+k-1,n+k) - \sum_{i=1}^{k-2} \mu(n+i,n+k) \\ &= \sum_{i=1}^{k-2} \mu(n+i,n+k)\end{aligned}$$

we can deduce inductively that $\mu(n,n+k) = 0$ for all $k \ge 2$. Therefore the Möbius Inversion Formula for the poset $\mathbb{N}_0$ states that for any functions $f, g\colon \mathbb{N}_0 \to \mathbb{R}$

$$f(n) = \sum_{i=0}^{n} g(i) \quad \text{if and only if} \quad \Big(f(0) = g(0) \text{ and } g(n) = f(n) - f(n-1)\Big)$$

for every $n \in \mathbb{N}$, which can be interpreted as a discrete version of the Fundamental Theorem of Calculus.

You will prove as an exercise that the Principle of Inclusion-Exclusion is also a specialization of the Möbius Inversion Formula. The following proposition will be useful to find an explicit formula for the Möbius function of a Boolean algebra.

Proposition 5. *Let P and Q be locally finite poset, and let $P \times Q$ denote the direct product poset of P and Q; that is, $(r,s) \le (r',s')$ in $P \times Q$ if and only if $r \le r'$ in P and $s \le s'$ in Q. Then $P \times Q$ is locally finite and for all $(r,s), (r',s') \in P \times Q$ with $(r,s) \le (r',s')$,*

$$\mu_{P\times Q}((r,s),(r',s')) = \mu_P(r,r')\mu_Q(s,s').$$

Proof. Exercise 4. □

We can inductively extend Proposition 5 to a finite product $P := P_1 \times \cdots \times P_n$ of locally finite posets $P_1, \dots, P_n$ to write μ_P as the product of the Möbius functions $\mu_{P_1}, \dots, \mu_{P_n}$. As an application of Proposition 5, let us find an explicit formula for the Möbius function of the Boolean algebra B_n.

Example 6. The Möbius function μ of the two-element chain $\{0,1\}$ with $0 < 1$ has values $\mu(0,0) = \mu(1,1) = 1$ and $\mu(0,1) = -1$. Now fix $n \in \mathbb{N}$, and observe that after identifying any $S \subseteq [n]$ with the vector $v_S := (v_S(1), \dots, v_S(n)) \in \{0,1\}^n$, where $v_S(i) = 1$ if and only if $i \in S$, we can think of the poset B_n as the product poset $\{0,1\}^n$. Therefore it follows from Proposition 5 that for any $S, T \subseteq [n]$ with $S \subseteq T$,

$$\mu_{B_n}(S,T) = \mu_{B_n}(v_S, v_T) = \prod_{i=1}^{n} \mu(v_S(i), v_T(i)) = (-1)^{|T \setminus S|}.$$

Practice Exercises

Exercise 1. *Let P be a locally finite poset, and let $f_1, \dots, f_k$ be elements in $A(P)$. Prove that*

$$(f_1 f_2 \dots f_k)(r,s) = \sum_{r = r_0 \le r_1 \le \cdots \le r_k = s} f_1(r_0, r_1) f_2(r_1, r_2) \cdots f_k(r_{k-1}, r_k).$$

Exercise 2. *Let P be a finite poset with $\hat{0}$ and $\hat{1}$, and let μ be the Möbius function of P. Prove that*

$$\sum_{\hat{0} = r_0 < r_1 < \cdots < r_k = \hat{1}} (-1)^k \mu(r_0, r_1) \mu(r_1, r_2) \cdots f_k(r_{k-1}, r_k) = 1.$$

Exercise 3. *Let P be a finite poset with $\hat{0}$ and $\hat{1}$, and let μ be the Möbius function of P. Prove that*

$$\sum_{r \le s} \mu(r,s) = 1.$$

Exercise 4. *Prove Proposition 5.*

Exercise 5. *Deduce the Principle of Inclusion-Exclusion as a specialization of the Möbius Inversion Formula for the Boolean algebra B_n.*

References

[1] R. P. Stanley: *Enumerative Combinatorics, Volume 1* (second edition), Cambridge Studies in Advanced Mathematics, Vol. 49, Cambridge University Press, New York, 2012.

Department of Mathematics, MIT, Cambridge, MA 02139
Email address: `fgotti@mit.edu`