MIT 18.211: COMBINATORIAL ANALYSIS

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Lecture 37: Intro to Lattices

In this lecture, we will give a brief introduction to lattices, which are posets where any finite subset of elements has both an infimum and a supremum. We proceed to formalize this notion. Let P be a poset, and let S be a nonempty subset of P. An element $r \in P$ is called a *lower bound* (resp., an *upper bound*) of S is $r \le s$ (resp., $s \le r$) for every $s \in S$. If r is a lower bound (resp., an upper bound) and $r' \le r$ (resp., $r \le r'$) for any other upper bound (resp., lower bound) r' of S, then we call r a *meet* (resp., *join*) of S. One can easily show that every subset of P has at most one meet, and the same observation holds for joins. We let $\wedge S$ and $\vee S$ denote the meet and the join of S, respectively (when they exists, of course). If $S = \{s_1, \dots, s_n\}$, then we often write $s_1 \wedge \cdots \wedge s_n$ (resp., $s_1 \vee \cdots \vee s_n$) instead of $\wedge S$ (resp., $\vee S$).

Definition 1. A poset L is called a *meet-semilattice* (resp., a *join-semilattice*) if every nonempty finite subset of L has a meet (resp., a join). A poset that is both a meet-semilattice and a join-semilattice is called a *lattice*.

A poset L is a meet-semilattice (resp., a join-semilattice) if and only if every two elements of L have a meet (resp., a join). The nontrivial part of this observation follows by induction (see Exercise 2).

For a poset P, if $\wedge P$ exists, then we denote it by $\hat{0}$, and if $\vee P$ exists, then we denote it by $\hat{1}$. In this lecture, we are specially interested in finite lattices. Observe that a finite lattice L has always $\hat{0}$ and $\hat{1}$, which are the only minimal and maximal elements of L, respectively.

Proposition 2. *Every finite meet-semilattice (resp., join-semilattice) with* $\hat{1}$ *(resp.,* $\hat{0}$*) is a lattice.*

Proof. Let L be a meet-semilattice with $\hat{1}$. Proving that L is a lattice amounts to arguing that any two elements $r, s \in L$ have a join. To do so, consider the set

$$S := \{x \in L \mid r \le x \ \text{ and } \ s \le x\}.$$

Since L is a finite poset with $\hat{1}$, we see that S is a finite nonempty subset of L. As L is a meet-semilattice, S has a meet $t \in L$. We claim that t is the join of r and s. Since r is a lower bound of S, we see that $r \le t$. Similarly, $s \le t$. Therefore t is an upper bound for $\{r, s\}$. Now suppose that $t' \in L$ is an upper bound for $\{r, s\}$. In this case, $t' \in S$ and, as a result, $t \le t'$. Hence t is the join of r and s in L, and we can conclude that L is a lattice. We can mimic the given arguments to prove that L is a lattice under the assumption that L is a join-semilattice with $\hat{0}$. $\square$

Observe that the notions of a meet and a join are generalizations of the respective notions of an infimum and a supremum in the real line. In general, every chain P is an example of a lattice, where $\wedge S = \min S$ and $\vee S = \max S$ for any nonempty finite subset S of P. In particular, $\boldsymbol{n}$ is a finite lattice. Here are further examples of lattices.

Example 3. For every $n \in \mathbb{N}$, the poset B_n is a lattice, where meets and joins are respectively given by intersections and unions of sets. We often call B_n a *Boolean lattice/algebra*.

Example 4. In the same way, for any prime-power q and $n \in \mathbb{N}$, the poset $B_n(q)$ is a lattice, where the meet of two subspaces of $\mathbb{F}_p^n$ is their intersection and the join of two subspaces of $\mathbb{F}_p^n$ is the subspace generated by their union.

Example 5. For every $n \in \mathbb{N}$, the poset Π_n is a lattice. For partitions $\sigma = \{A_1, \dots, A_k\}$ and $\tau = \{B_1, \dots, B_\ell\}$, the partition $\{A_i \cap B_j \mid i \in [k] \text{ and } j \in [\ell]\}$ can be checked to be the meet of σ and τ in Π_n. Therefore Π_n is a meet-semilattice. Since Π_n is finite with $\hat{1} = \{[n]\}$, it follows from Proposition 2 that Π_n is a lattice.

Modularity. A lattice L is called *upper semimodular* if for all $r, s \in L$, the fact that r and s both cover $r \wedge s$ implies that $r \vee s$ covers both r and s.

Theorem 6. *For a finite lattice L, the following conditions are equivalent.*

(a) *L is graded and $\rho(r)+\rho(s) \ge \rho(r \wedge s)+\rho(r \vee s)$ for all $r, s \in L$, where $\rho\colon L \to \mathbb{N}_0$ is the rank function of L.*

(b) *L is an upper semimodular lattice.*

Proof. (a) $\Rightarrow$ (b): Take $r, s \in L$ such that r and s both cover $r \wedge s$. Clearly, $r \neq s$ and both equalities $\rho(r) = \rho(r \wedge s) + 1$ and $\rho(s) = \rho(r \wedge s) + 1$ hold. Since r and s both cover $r \wedge s$, the elements r and s are not comparable, and so $\rho(r \vee s) \ge \rho(r) + 1$ and $\rho(r \vee s) \ge \rho(s) + 1$. In addition, it follows from the inequality of condition (a) that

$$\rho(r \vee s) \le \rho(r) + (\rho(s) - \rho(r \wedge s)) = \rho(r) + 1.$$

Thus, $\rho(r \vee s) = \rho(r) + 1$, which implies that $r \vee s$ covers r. We can similarly show that $r \vee s$ also covers s.

(b) $\Rightarrow$ (a): Assume, towards a contradiction, that L is not graded. Then there must be an interval $[r, s]$ of L with mimimum length that is not graded. Let

$$r = t_0 \lessdot t_1 \lessdot \cdots \lessdot t_m = s \quad \text{and} \quad r = t'_0 \lessdot t'_1 \lessdot \cdots \lessdot t'_n = s$$

be two maximal chains in $[r, s]$ such that $m \neq n$. Since both t_1 and t_1' cover r, it follows that $t_1 \vee t_1'$ cover both t_1 and t_1'. As s is an upper bound for $\{t_1, t_1'\}$, we see that $t_1 \vee t_1' \leq s$. Let $t_1 \vee t_1' = u_0 \lessdot u_1 \lessdot \cdots \lessdot u_\ell = s$ be a saturated chain. Then

$$t_1 \lessdot u_0 \lessdot u_1 \lessdot \cdots \lessdot u_\ell = s \quad \text{and} \quad t_1 \lessdot \cdots \lessdot t_m = s$$

are both maximal chains in the interval $[t_1, s]$. By the minimality of $[r, s]$, the interval $[t_1, s]$ is graded, which implies that $m = \ell + 2$. In a similar way, $[t_1', s]$ is a graded interval, and so the fact that

$$t_1' \lessdot u_0 \lessdot u_1 \lessdot \cdots \lessdot u_\ell = s \quad \text{and} \quad t_1' \lessdot \cdots \lessdot t_n' = s$$

are both maximal chains in $[t_1', s]$ guarantees that $n = \ell + 2$. However, this contradicts that $m \neq n$. Hence L must be graded.

Finally assume, by way of contradiction, that there exist $r, s \in L$ such that

$$\rho(r) + \rho(s) < \rho(r \wedge s) + \rho(r \vee s). \tag{0.1}$$

Among all such pairs of elements satisfying (0.1), assume that we have taken r and s minimizing the length of the interval $[r \wedge s, r \vee s]$ and then the sum $\rho(r) + \rho(s)$. The inequality (0.1), along with condition (b), ensures that not both r and s can cover $r \wedge s$. Then we can assume that there is an $s' \in L$ such that $r \wedge s < s' < s$. It is clear that $r \wedge s' = r \wedge s$. In addition, it follows from the minimality of both the length of $[r, s]$ and the sum $\rho(r) + \rho(s)$ that

$$\rho(r) + \rho(s') \geq \rho(r \wedge s') + \rho(r \vee s'). \tag{0.2}$$

Now we can bound $\rho(r \vee s')$ from above by using (0.2) and then use the inequality (0.1) to obtain the following:

$$\begin{aligned}
\rho(s) + \rho(r \vee s') &\leq \rho(s) + (\rho(r) + \rho(s') - \rho(r \wedge s')) \\
&= \rho(s') + (\rho(r) + \rho(s) - \rho(r \wedge s)) \\
&< \rho(s') + \rho(r \vee s) \\
&\leq \rho(s \wedge (r \vee s')) + \rho(s \vee (r \vee s')).
\end{aligned}$$

Therefore, after setting $r' := r \vee s'$, we obtain that $\rho(s) + \rho(r') < \rho(s \wedge r') + \rho(s \vee r')$. However, this inequality contradicts the minimality conditions satisfied by r and s; indeed, observe that the length of $[s \wedge r', s \vee r']$ is at most the length of $[r \wedge s, r \vee s]$ and $\rho(s) + \rho(r') < \rho(r) + \rho(s)$. □

The *dual poset* of a poset P is the poset $P^* := (P, \leq^*)$ such that $r \leq^* s$ in P^* if and only if $r \geq s$ in P. We say that a lattice L is *modular* if both L and its dual poset L^* are upper semimodular. It follows from Theorem 6 that a finite lattice L is modular if and only if L is graded and $\rho(r) + \rho(s) = \rho(r \wedge s) + \rho(r \vee s)$ for all $r, s \in L$, where ρ is the rank function of L.

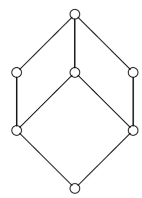
Example 7. For every j and k in the graded poset $\boldsymbol{n}$,

$$\rho(j \wedge k) + \rho(j \vee k) = \min\{j, k\} + \max\{j, k\} = j + k.$$

Therefore $\boldsymbol{n}$ is a modular lattice.

Example 8. We have seen before that for every $n \in \mathbb{N}$, the Boolean lattice B_n is a graded poset whose rank function $\rho\colon B_n \to \mathbb{N}_0$ is given by $\rho(A) = |A|$. Therefore the fact that $|A| + |B| = |A \cap B| + |A \cup B|$ for all $A, B \in B_n$ implies that B_n is a modular lattice.

Example 9. The lattice whose Hasse diagram is shown below is an upper semimodular finite lattice that is not modular (check this!).



Atomic and Geometric Lattices. Let P be a poset with $\hat{0}$. An element of P is called an *atom* if it covers $\hat{0}$. A join-semilattice P with $\hat{0}$ is called *atomic* if every element of P different from $\hat{0}$ is the join of finitely many atoms. In a similar way, one can define *coatoms* and *coatomic* posets.

Example 10. Fix $n \in \mathbb{N}$. Observe that if $n = 1$, then the lattice $\boldsymbol{n}$ has no atoms but still it is atomic. Assume now that $n \geq 2$. Then the only atom of $\boldsymbol{n}$ is 2, and so $\boldsymbol{n}$ is atomic if and only if $n = 2$; indeed, if $n \geq 3$, then 3 is an element of $\boldsymbol{n}$ that is not the join of atoms.

Definition 11. An upper semimodular lattice that is atomic is called a *geometric lattice.*

The upper semimodular lattice in Example 9 is not atomic, and so it is not geometric.

Example 12. We have observed before that the Boolean lattice B_n is modular. In addition, note that the atoms of B_n are precisely the singletons, that is, the 1-subsets of $[n]$. Since every subset of $[n]$ is the union of singletons, B_n is atomic and, therefore, a geometric lattice.

Example 13. Fix $n \in \mathbb{N}$, and consider the lattice Π_n. We have seen before that Π_n is a graded poset with rank function given by $\rho(\sigma) = n - |\sigma|$, where $|\sigma|$ is the number of blocks of the partition σ. It is clear that the atoms of Π_n are the partitions of $[n]$ having only one non-singleton block and this block having size 2. Therefore it is clear that every element of Π_n is the join of atoms. Hence Π_n is atomic. It turns out that Π_n is upper semimodular and, therefore, a geometric lattice. However, Π_n is not a modular lattice. The last two statements are left as an exercise (Exercise 5).

Practice Exercises

Exercise 1. *Let P be a poset, and let S be a subset of P. Prove that S has at most a meet (or a join) in P.*

Exercise 2. *Prove that a poset L is a meet-semilattice (resp., a join-semilattice) if and only if any pair of elements of L has a meet (resp., a join).*

Exercise 3. *A nonempty subset S of a lattice L is called a* ***sublattice*** *if for all $r, s \in S$ both $r \wedge s$ and $r \vee s$ belong to S. Prove that every sublattice of a modular lattice is also a modular lattice.*

Exercise 4. *Let n be a positive integer and let q be a positive power of a prime. Prove that the lattice $B_n(q)$ is both modular and geometric.*

Exercise 5. *Prove that the lattice Π_n is upper semimodular but not modular.*

References

[1] R. P. Stanley: *Enumerative Combinatorics, Volume 1* (second edition), Cambridge Studies in Advanced Mathematics, Vol. 49, Cambridge University Press, New York, 2012.

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