Lecture 35: Intro to Posets

In this lecture we take a first look at partially ordered sets.

Definition 1. A pair \((P, \leq)\), where \(P\) is a nonempty set and \(\leq\) is a relation on \(P\), is called a partially ordered set or poset provided that the following conditions hold.

1. Reflexivity: \(x \leq x\) for every \(x \in P\).
2. Antisymmetry: for every \(x, y \in P\), if \(x \leq y\) and \(y \leq x\), then \(x = y\).
3. Transitivity: for every \(x, y, z \in P\), if \(x \leq y\) and \(y \leq z\), then \(x \leq z\).

When there seems to be no risk of ambiguity, we write \(P\) instead of \((P, \leq)\). Let \(P\) be a poset. We say that \(P\) is a totally ordered poset or a chain if for all \(x, y \in P\) either \(x \leq y\) or \(y \leq x\), that is, any two elements of \(P\) are comparable. For instance, the real line \(\mathbb{R}\) with the standard order relation is a chain. The set \([n]\) with the standard order relation is a finite chain, which we denote here by \(n\). It is clear that a subset \(S\) of a poset \(P\) is also a poset under the order relation it inherits from \(P\); the poset \(S\) is called a subposet of \(P\). For instance, \(\mathbb{Z}\) and \(\mathbb{Q}\) are subposets of \(\mathbb{R}\) under the standard order relation. Let us show less trivial examples.

Example 2. The collection of all subsets of \([n]\) is a poset under the inclusion operation \(\subseteq\), which is denoted by \(B_n\). Observe that \(B_n\) is not a chain when \(n \geq 2\) as the subsets \(\{1\}\) and \(\{2\}\) of \([n]\) are not comparable.

Example 3. Let \(\Pi_n\) denote the poset consisting of all partitions of \([n]\) under the refinement relation: two partitions \(\pi_1, \pi_2 \in \Pi_n\) satisfy \(\pi_1 \leq \pi_2\) if and only if every block of \(\pi_1\) is contained in \(\pi_2\). The poset \(\Pi_n\) is a chain if and only if \(n \leq 2\) because when \(n \geq 3\), the partitions \(\{\{1, 2\}, \{3\}, [n] \setminus \{1, 2, 3\}\}\) and \(\{\{1\}, \{2, 3\}, [n] \setminus \{1, 2, 3\}\}\) are not comparable.

Example 4. Let \(n\) be a positive integer and let \(q\) be a positive power of a prime. Then let \(V := \mathbb{F}_q^n\) be the \(n\)-dimensional vector space over \(\mathbb{F}_q\). Now let \(B_n(q)\) denote the poset of all subspaces of \(V\) under inclusion. If \(n \geq 2\), then the subspaces of \(V\) generated by the canonical vectors \(e_1\) and \(e_2\) are not comparable, and so \(B_n(q)\) is not a chain.
An element $x \in P$ is called minimal (resp., maximal) if there is not an element in $P$ strictly smaller (resp., larger) than $x$. An element $x \in P$ is called minimum (resp., maximum) if $x \leq y$ (resp., $x \geq y$) for all $y \in P$.

For $x, y \in P$ with $x \leq y$, we call $[x, y] := \{z \in P \mid x \leq z \leq y\}$ an interval of $P$. If every interval of $P$ is finite we say that $P$ is locally finite. Clearly, every finite poset is locally finite. When the interval $[x, y]$ has size 2, we say that $y$ covers $x$ and write $x \lessdot y$.

The Hasse diagram of a finite poset $P$ is a graph whose vertices are the element of $P$ and whose edges are given by the cover relations satisfying that for all $s, t \in P$ with $s < t$ the element $s$ is drawn (horizontally) strictly below $t$. Here is the Hasse diagram of $B_3$:

A chain of $P$ is a subposet of $P$ that is a chain as a poset. The length of a chain $C$ of $P$ is $\ell(C) := |C| - 1$. A chain $x_0 < x_1 < \cdots < x_\ell$ is saturated if $x_{n-1} < x_n$ for every $n \in [\ell]$. A chain of $P$ is maximal if it is not strictly contained in any other chain. Observe that every maximal chain is saturated.

**Definition 5.** A finite poset $P$ is called graded of rank $\ell$ for some $\ell \in \mathbb{N}_0$ if every maximal chain of $P$ has length $\ell$.

For a poset $P$, we say that a function $\rho: P \to \mathbb{N}_0$ is a rank function if $\rho$ satisfies the following conditions:

1. $\rho(m) = 0$ for every minimal element $m \in P$, and
2. $\rho(y) = \rho(x) + 1$ if $x, y \in P$ and $x \lessdot y$.

It turns out that every finite graded poset has a rank function.

**Proposition 6.** Every graded poset has a unique rank function.

**Proof.** Let $P$ be a graded poset. Define $\rho: P \to \mathbb{N}_0$ by setting $\rho(y) := |\{x \in C \mid x < y\}|$, where $C$ is a maximal chain of $P$ containing $y$. Let us check that $\rho(y)$ does not depend on the chosen maximal chain $C$ containing $y$. To do so, let $C'$ be another maximal chain of $P$ containing $y$. It suffices to show that the sets

$$C_y := \{x \in C \mid x < y\} \quad \text{and} \quad C'_y := \{x \in C' \mid x < y\}$$

have the same cardinality. If $|C_y| \neq |C'_y|$, say $|C_y| > |C'_y|$, then the chain $C_y \cup (C' \setminus C_y)$ is a chain of $P$ whose length is larger than the length of $C'$, which contradicts the fact that any two maximal chains of $P$ have the same length. Therefore $|C_y| = |C'_y|$. Hence $\rho$ is well defined.
It follows directly from the definition of \( \rho \) that \( \rho(m) = 0 \) for every minimal element \( m \in P \). Now, suppose that \( x, y \in P \) with \( x < y \), and take a maximal chain \( C \) containing both \( x \) and \( y \). Then we see that
\[
\rho(y) = \left| \{ c \in C \mid c < x \} \cup \{ x \} \right| = \left| \{ c \in C \mid c < x \} \right| + 1 = \rho(x) + 1.
\]
As a result, \( \rho \) is a rank function.

The uniqueness of the rank function is left as an exercise. \( \square \)

**Corollary 7.** If \( P \) is a graded poset, then every interval of \( P \) is also a graded poset.

Let \( P \) be a graded poset with rank function \( \rho: P \to \mathbb{N}_0 \). The rank of \( a \in P \) is \( \rho(a) \). The rank-generating function of \( P \) is the polynomial
\[
f(P, X) := \sum_{a \in P} X^{\rho(a)} = \sum_{k=0}^{\ell} c_k X^k,
\]
where \( \ell \) is the rank of \( P \) and \( c_k \) is the number of elements of \( P \) with rank \( k \) (for every \( k \in [0, \ell] \)). Let us further discuss the examples we mentioned at the beginning of the lecture.

**Example 8.** Every finite chain is clearly a graded poset. In particular, \( n \) is a graded poset of rank \( n - 1 \) with rank function given by \( \rho(k) = k - 1 \). The poset \( n \) has exactly one element of rank \( k \) for every \( k \in [0, n - 1] \). Therefore the rank-generating function of \( n \) is
\[
f(n, X) = \sum_{k=0}^{n-1} X^k = \frac{1 - X^n}{1 - X}.
\]

**Example 9.** For \( n \in \mathbb{N} \), consider the poset \( B_n \). Observe that \( S \) is covered by \( T \) in \( B_n \) if and only if \( T = S \cup \{ t \} \) for some \( t \in T \setminus S \). Therefore any maximal chain in \( B_n \) has length \( n \), and so \( B_n \) is a graded poset of rank \( n \) whose rank function satisfies \( \rho(S) = |S| \). In addition, the rank-generating function of \( B_n \) is
\[
f(B_n, X) = \sum_{k=0}^{n} \binom{n}{k} X^k = (1 + X)^n.
\]

**Example 10.** Now consider the poset \( \Pi_n \) for \( n \in \mathbb{N} \). It is not hard to verify that for all \( \pi_1, \pi_2 \in \Pi_n \), the relation \( \pi_1 < \pi_2 \) holds if and only if \( \pi_2 \) can be obtained from \( \pi_1 \) by joining two distinct blocks of the latter. Therefore every maximal chain of \( \Pi_n \) has length \( n - 1 \), and so \( \Pi_n \) is a graded poset of rank \( n - 1 \) whose rank function satisfies \( \rho(\pi) = n - |\pi| \). Then the rank-generating function of \( \Pi_n \) is
\[
f(\Pi_n, X) = \sum_{k=0}^{n-1} S(n, n - k) X^k.
\]
Example 11. Fix \( p \in \mathbb{P} \) and \( n \in \mathbb{N} \), and then consider the poset \( B_n(q) \), where \( q \) is a positive power of \( p \). Two subspaces \( W_1 \) and \( W_2 \) of \( \mathbb{F}^n_q \) satisfy \( W_1 \prec W_2 \) if and only if \( W_1 \subseteq W_2 \) and \( \dim W_2 = 1 + \dim W_1 \). Then \( B_n(q) \) is graded of rank \( n \) whose rank function is given by \( \rho(W) = \dim W \). Then the rank-generating function of \( B_n(q) \) is

\[
 f(B_n(q), X) = \sum_{k=0}^{n} \binom{n}{k} X^k.
\]

A subset \( A \) of a poset \( P \) is called an antichain if no two elements of \( A \) are comparable in \( P \). A chain cover of \( P \) is a collection of finitely many chains of \( P \) whose union is \( P \). It turns out that the minimum number of chains required to have a chain cover of \( P \) coincides with the size of a maximum-size antichain of \( P \).

Theorem 12 (Dilworth’s Theorem). Let \( P \) be a finite poset. Then the number of chains in a minimum-size chain cover of \( P \) equals the size of a maximum-size antichain of \( P \).

Proof. If \( A \) is an antichain of \( P \) of maximum size and \( m \) is the size of any minimum-size chain cover of \( P \), then the fact that every chain of \( P \) intersects \( A \) in at most one element ensures that \( |A| \leq m \). Thus, it suffices to show that \( P \) can be covered using \( |A| \) chains, where \( A \) is a maximum-size antichain of \( P \). We proceed by induction on the size of \( P \). The case \( |P| = 1 \) is straightforward. Now assume that \( n := |P| \geq 2 \) and that the statement of the result we are willing to establish holds for every poset whose size is strictly less than \( |P| \). We split the rest of the proof into two cases.

Case 1: \( P \) has a maximum-size antichain \( A \) that contains an element that is not minimal and an element that is not maximal. Consider now the sets

\[
 L := \{ x \in P \mid x \leq a \text{ for some } a \in A \}
\]

and

\[
 U := \{ x \in P \mid x \geq a \text{ for some } a \in A \}.
\]

It is clear that \( L \cup U = P \) and \( L \cap U = A \). In addition, from the fact that \( A \) contains an element that is not minimal, we deduce that \( L \) is nonempty. In a similar way, we deduce that \( U \) is nonempty. Therefore both \( L \) and \( U \) are subposets of \( P \) satisfying \( |L| < |P| \) and \( |U| < |P| \). Our induction hypothesis guarantees that each of the subposets \( L \) and \( U \) has a chain cover consisting of \( |A| \) chains. Observe that each \( a \in A \) belongs to exactly one chain \( L_a \) in the chain cover of \( L \) and one chain \( U_a \) in the chain cover of \( U \). Since, for each \( a \in A \), the set \( L_a \cup U_a \) is a chain of \( P \), it follows that \( \{ L_a \cup U_a \mid a \in A \} \) is a chain cover of \( P \) consisting of \( |A| \) chains.

Case 2: Each maximum-size antichain of \( P \) consists of either minimal elements of \( P \) or maximal elements of \( P \). Take \( x, y \in P \) with \( x \leq y \) such that \( x \) is minimal and \( y \) is maximal (perhaps \( x = y \)). Since every antichain of the subposet \( P' := P \setminus \{ x, y \} \) is also an antichain of \( P \), every maximum-size antichain of \( P' \) must contain \( |A| - 1 \) elements (as such a maximum-size antichain of \( P' \) cannot contain all maximal (or minimal) elements of \( P \)). Now our induction hypothesis allows us to pick a chain cover \( \mathcal{C} \) of \( P' \).
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consisting of $|A| - 1$ chains. Then $\mathcal{C} \cup \{x \leq y\}$ is a chain cover of $P$ consisting of $|A|$ chains. □

practice exercises

exercise 1. draw the hasse diagrams of all sixteen posets of size 4.

exercise 2. for each $n \in \mathbb{N}$, how many maximal chains does $B_n$ have?

exercise 3. prove that every poset has at most one rank function.

references


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