MIT 18.211: COMBINATORIAL ANALYSIS

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Lecture 35: Intro to Posets

In this lecture we take a first look at partially ordered sets.

Definition 1. A pair $(P, \leq)$, where P is a nonempty set and $\leq$ is a relation on P, is called a *partially ordered set* or *poset* provided that the following conditions hold.

(1) Reflexivity: $x \leq x$ for every $x \in P$.
(2) Antisymmetry: for every $x, y \in P$, if $x \leq y$ and $y \leq x$, then $x = y$.
(3) Transitivity: for every $x, y, z \in P$, if $x \leq y$ and $y \leq z$, then $x \leq z$.

When there seems to be no risk of ambiguity, we write P instead of $(P, \leq)$. Let P be a poset. We say that P is a *totally ordered poset* or a *chain* if for all $x, y \in P$ either $x \leq y$ or $y \leq x$, that is, any two elements of P are comparable. For instance, the real line $\mathbb{R}$ with the standard order relation is a chain. The set $[n]$ with the standard order relation is a finite chain, which we denote here by $\boldsymbol{n}$. It is clear that a subset S of a poset P is also a poset under the order relation it inherits from P; the poset S is called a *subposet* of P. For instance, $\mathbb{Z}$ and $\mathbb{Q}$ are subposets of $\mathbb{R}$ under the standard order relation. Let us show less trivial examples.

Example 2. The collection of all subsets of $[n]$ is a poset under the inclusion operation $\subseteq$, which is denoted by B_n. Observe that B_n is not a chain when $n \geq 2$ as the subsets $\{1\}$ and $\{2\}$ of $[n]$ are not comparable.

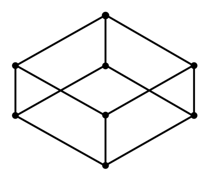
Example 3. Let Π_n denote the poset consisting of all partitions of $[n]$ under the refinement relation: two partitions $\pi_1, \pi_2 \in \Pi_n$ satisfy $\pi_1 \leq \pi_2$ if and only if every block of π_1 is contained in π_2. The poset Π_n is a chain if and only if $n \leq 2$ because when $n \geq 3$, the partitions $\{\{1,2\}, \{3\}, [n] \setminus \{1,2,3\}\}$ and $\{\{1\}, \{2,3\}, [n] \setminus \{1,2,3\}\}$ are not comparable.

Example 4. Let n be a positive integer and let q be a positive power of a prime. Then let $V := \mathbb{F}_q^n$ be the n-dimensional vector space over $\mathbb{F}_q$. Now let $B_n(q)$ denote the poset of all subspaces of V under inclusion. If $n \geq 2$, then the subspaces of V generated by the canonical vectors e_1 and e_2 are not comparable, and so $B_n(q)$ is not a chain.

An element $x \in P$ is called *minimal* (resp., *maximal*) if there is not an element in P strictly smaller (resp., larger) than x. An element $x \in P$ is called *minimum* (resp., *maximum*) if $x \leq y$ (resp., $x \geq y$) for all $y \in P$.

For $x, y \in P$ with $x \leq y$, we call $[x, y] := \{z \in P \mid x \leq z \leq y\}$ an *interval* of P. If every interval of P is finite we say that P is *locally finite*. Clearly, every finite poset is locally finite. When the interval $[x, y]$ has size 2, we say that y *covers* x and write $x \lessdot y$.

The *Hasse diagram* of a finite poset P is a graph whose vertices are the element of P and whose edges are given by the cover relations satisfying that for all $s, t \in P$ with $s < t$ the element s is drawn (horizontally) strictly below t. Here is the Hasse diagram of B_3:



A *chain of* P is a subposet of P that is a chain as a poset. The *length* of a chain C of P is $\ell(C) := |C| - 1$. A chain $x_0 < x_1 < \cdots < x_\ell$ is *saturated* if $x_{n-1} \lessdot x_n$ for every $n \in [\ell]$. A chain of P is *maximal* if it is not strictly contained in any other chain. Observe that every maximal chain is saturated.

Definition 5. A finite poset P is called *graded* of *rank* ℓ for some $\ell \in \mathbb{N}_0$ if every maximal chain of P has length ℓ.

For a poset P, we say that a function $\rho \colon P \to \mathbb{N}_0$ is a *rank function* if ρ satisfies the following conditions:

(1) $\rho(m) = 0$ for every minimal element $m \in P$, and
(2) $\rho(y) = \rho(x) + 1$ if $x, y \in P$ and $x \lessdot y$.

It turns out that every finite graded poset has a rank function.

Proposition 6. *Every graded poset has a unique rank function.*

Proof. Let P be a graded poset. Define $\rho \colon P \to \mathbb{N}_0$ by setting $\rho(y) := |\{x \in C \mid x < y\}|$, where C is a maximal chain of P containing y. Let us check that $\rho(y)$ does not depend on the chosen maximal chain C containing y. To do so, let C' be another maximal chain of P containing y. It suffices to show that the sets

$$C_y := \{x \in C \mid x < y\} \quad \text{and} \quad C'_y := \{x \in C' \mid x < y\}$$

have the same cardinality. If $|C_y| \neq |C'_y|$, say $|C_y| > |C'_y|$, then the chain $C_y \cup (C' \setminus C_y)$ is a chain of P whose length is larger than the length of C', which contradicts the fact that any two maximal chains of P have the same length. Therefore $|C_y| = |C'_y|$. Hence ρ is well defined.

It follows directly from the definition of ρ that $\rho(m) = 0$ for every minimal element $m \in P$. Now, suppose that $x, y \in P$ with $x \lessdot y$, and take a maximal chain C containing both x and y. Then we see that

$$\rho(y) = |\{c \in C \mid c < x\} \sqcup \{x\}| = |\{c \in C \mid c < x\}| + 1 = \rho(x) + 1.$$

As a result, ρ is a rank function.

The uniqueness of the rank function is left as an exercise. □

Corollary 7. *If P is a graded poset, then every interval of P is also a graded poset.*

Let P be a graded poset with rank function $\rho\colon P \to \mathbb{N}_0$. The *rank* of $a \in P$ is $\rho(a)$. The *rank-generating function* of P is the polynomial

$$f(P, X) := \sum_{a \in P} X^{\rho(a)} = \sum_{k=0}^{\ell} c_k X^k,$$

where ℓ is the rank of P and c_k is the number of elements of P with rank k (for every $k \in [\![0, \ell]\!]$). Let us further discuss the examples we mentioned at the beginning of the lecture.

Example 8. Every finite chain is clearly a graded poset. In particular, $\boldsymbol{n}$ is a graded poset of rank $n-1$ with rank function given by $\rho(k) = k - 1$. The poset $\boldsymbol{n}$ has exactly one element of rank k for every $k \in [\![0, n-1]\!]$. Therefore the rank-generating function of $\boldsymbol{n}$ is

$$f(\boldsymbol{n}, X) = \sum_{k=0}^{n-1} X^k = \frac{1 - X^n}{1 - X}.$$

Example 9. For $n \in \mathbb{N}$, consider the poset B_n. Observe that S is covered by T in B_n if and only if $T = S \cup \{t\}$ for some $t \in T \setminus S$. Therefore any maximal chain in B_n has length n, and so B_n is a graded poset of rank n whose rank function satisfies $\rho(S) = |S|$. In addition, the rank-generating function of B_n is

$$f(B_n, X) = \sum_{k=0}^{n} \binom{n}{k} X^k = (1 + X)^n.$$

Example 10. Now consider the poset Π_n for $n \in \mathbb{N}$. It is not hard to verify that for all $\pi_1, \pi_2 \in \Pi_n$, the relation $\pi_1 \lessdot \pi_2$ holds if and only if π_2 can be obtained from π_1 by joining two distinct blocks of the latter. Therefore every maximal chain of Π_n has length $n - 1$, and so Π_n is a graded poset of rank $n - 1$ whose rank function satisfies $\rho(\pi) = n - |\pi|$. Then the rank-generating function of Π_n is

$$f(\Pi_n, X) = \sum_{k=0}^{n-1} S(n, n-k) X^k.$$

Example 11. Fix $p \in \mathbb{P}$ and $n \in \mathbb{N}$, and then consider the poset $B_n(q)$, where q is a positive power of p. Two subspaces W_1 and W_2 of $\mathbb{F}_q^n$ satisfy $W_1 \lessdot W_2$ if and only if $W_1 \subseteq W_2$ and $\dim W_2 = 1 + \dim W_1$. Then $B_n(q)$ is graded of rank n whose rank function is given by $\rho(W) = \dim W$. Then the rank-generating function of $B_n(q)$ is

$$f(B_n(q), X) = \sum_{k=0}^{n} \binom{n}{k}_q X^k.$$

A subset A of a poset P is called an *antichain* if no two elements of A are comparable in P. A *chain cover* of P is a collection of finitely many chains of P whose union is P. It turns out that the minimum number of chains required to have a chain cover of P coincides with the size of a maximum-size antichain of P.

Theorem 12 (Dilworth's Theorem)**.** *Let P be a finite poset. Then the number of chains in a minimum-size chain cover of P equals the size of a maximum-size antichain of P.*

Proof. If A is an antichain of P of maximum size and m is the size of any minimum-size chain cover of P, then the fact that every chain of P intersects A in at most one element ensures that $|A| \le m$. Thus, it suffices to show that P can be covered using $|A|$ chains, where A is a maximum-size antichain of P. We proceed by induction on the size of P. The case $|P| = 1$ is straightforward. Now assume that $n := |P| \ge 2$ and that the statement of the result we are willing to establish holds for every poset whose size is strictly less than $|P|$. We split the rest of the proof into two cases.

Case 1: P has a maximum-size antichain A that contains an element that is not minimal and an element that is not maximal. Consider now the sets

$$L := \{x \in P \mid x \le a \text{ for some } a \in A\}$$

and

$$U := \{x \in P \mid x \ge a \text{ for some } a \in A\}.$$

It is clear that $L \cup U = P$ and $L \cap U = A$. In addition, from the fact that A contains an element that is not minimal, we deduce that L is nonempty. In a similar way, we deduce that U is nonempty. Therefore both L and U are subposets of P satisfying $|L| < |P|$ and $|U| < |P|$. Our induction hypothesis guarantees that each of the subposets L and U has a chain cover consisting of $|A|$ chains. Observe that each $a \in A$ belongs to exactly one chain L_a in the chain cover of L and one chain U_a in the chain cover of U. Since, for each $a \in A$, the set $L_a \cup U_a$ is a chain of P, it follows that $\{L_a \cup U_a \mid a \in A\}$ is a chain cover of P consisting of $|A|$ chains.

Case 2: Each maximum-size antichain of P consists of either minimal elements of P or maximal elements of P. Take $x, y \in P$ with $x \le y$ such that x is minimal and y is maximal (perhaps $x = y$). Since every antichain of the subposet $P' := P \setminus \{x, y\}$ is also an antichain of P, every maximum-size antichain of P' must contain $|A| - 1$ elements (as such a maximum-size antichain of P' cannot contain all maximal (or minimal) elements of P). Now our induction hypothesis allows us to pick a chain cover $\mathscr{C}$ of P'

consisting of $|A|-1$ chains. Then $\mathscr{C} \cup \{x \le y\}$ is a chain cover of P consisting of $|A|$ chains. $\square$

Practice Exercises

Exercise 1. *Draw the Hasse diagrams of all sixteen posets of size* 4.

Exercise 2. *For each $n \in \mathbb{N}$, how many maximal chains does B_n have?*

Exercise 3. *Prove that every poset has at most one rank function.*

References

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