MIT 18.211: COMBINATORIAL ANALYSIS

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Lecture 34: Polytopes and Platonic Solids

A *half-space* of $\mathbb{R}^3$ is the set of all points satisfying the inequality $ax + by + cz \le d$ for prescribed constants $a, b, c, d \in \mathbb{R}$. For instance, the set $\{(x, y, z) \in \mathbb{R}^3 \mid z \ge 1\}$ is a half-space (observe that the inequality $z \ge 1$ is equivalent to $0x+0y+(-1)z \le -1$). A (3-dimensional) *polyhedron* is the intersection of finitely many half-spaces. For instance, the first octant of $\mathbb{R}^3$ is a polyhedron as it is the intersection of three half-spaces, namely, those determined by $x \ge 0$, $y \ge 0$, and $z \ge 0$. Observe that the first octant is a bounded polyhedron, in the sense that it is not contained inside any sphere (centered at the origin). There are polyhedra that are bounded. Indeed, we can cut a bounded piece T of the first octant if we intersect it with the half-space $x + y + z \le 1$:

$$T := \{(x, y, z) \in \mathbb{R}^3 \mid x, y, z \ge 0 \text{ and } x + y + z \le 1\}.$$

The bounded polyhedron T is called a *tetrahedron*.

Definition 1. A bounded polyhedron is called a *polytope*.

Every polytope is a convex solid whose boundary consists of convex polygons, which are called *faces*. The boundary of any face consists of sides and vertices, and these are respectively called *edges* and *vertices* of the given polytope. As for graphs, the degree of a vertex v of a polytope is the number of edges incident to v. Let P be a polytope. We make the following geometric observations.

Remark 2.

- The boundary of every face of P consists of at least 3 edges.
- The degree of every vertex of P is at least 3.

In this lecture, we are mainly interested in the relations among the numbers of faces, edges, and vertices of a polytope. As for planar graphs, we will let F, E, V denote the numbers of faces, edges, and vertices of P, respectively.

Example 3. We have seen that the tetrahedron T is a polytope. It has 4 faces, 6 edges, and 4 vertices. Therefore $V - E + F = 2$.

Example 4. The cube is also a polytope. We can describe the unit cube in terms of half-spaces as follows:

$$\{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq x, y, z \leq 1\}.$$

Observe that a cube has 6 faces, 12 edges, and 8 vertices, and so it follows that $V - E + F = 2$.

The fact that tetrahedra and cubes satisfy the identity $V - E + F = 2$ is not a coincidence. As it is the case for planar graphs, any polytope satisfies the Euler's formula. Although there is a natural way to think of the boundary of a polytope as a planar graph (in which case, the relation $V - E + F = 2$ is simply a special case of the Euler's theorem), we will make use of the geometry of polytopes to prove that they satisfy Euler's formula.

Theorem 5. *Let P be a (3-dimensional) polytope with F faces, E edges, and V vertices. Then $V - E + F = 2$.*

Proof. Imagine we are holding in our hands the polytope P (whose faces and interior are transparent), and we are in front of a huge blackboard. If we place one of our eyes very close to one of the faces of P and look at the blackboard through that face, then we will see a planar projection of P on the blackboard, which is a graph without edges crossing and whose convex hull (or biggest face) is the projection of the face we are looking through. We consider this projection P' of P as a planar graph whose vertices, edges, and faces are the visual projections of the vertices, edges, and faces of P, except that instead of considering the unbounded face of P', we consider the biggest face given by the convex hull of P' (that is, the projection of the face of P we are looking through).

Let S be the addition of all the interior angles of all the faces of P' (including the biggest face). Let $v_1, \dots, v_m$ be the vertices of the biggest face of P', and let $v'_1, \dots, v'_n$ be the rest of the vertices. Then adding first the interior angles at the vertices $v_1, \dots, v_m$, we obtain $2 \cdot 180(m - 2)$, where the factor 2 is due to the fact that, besides adding the angles of the smaller faces, which add up to $180(m - 2)$, we also have to add all the interior angles of the biggest face, which also add up to $180(m - 2)$. Adding now the angles at the interior vertices $v'_1, \dots, v'_n$, we obtain $360n$. Hence

$$S = 360(m - 2) + 360n = 360(m + n - 2) = 360(V - 2).$$

Observe now that if $E_1, \dots, E_F$ are the numbers of sides of the F faces of P', then $\sum_{i=1}^{F} E_i = 2E$ because every edge is in the boundary of exactly two faces. Then we can find another expression for S as follows:

$$S = \sum_{i=1}^{F} 180(E_i - 2) = 180\Big(\sum_{i=1}^{F} E_i\Big) - 360F = 360(E - F).$$

Therefore $360(V - 2) = S = 360(E - F)$, which implies that $V - E + F = 2$. □

The following proposition is often a helpful tool.

Proposition 6. *Let P be a polytope. If $E_1, \dots, E_F$ are the numbers of edges of the F faces of P and $d_1, \dots, d_V$ are the degrees of the V vertices of P, then*

$$\sum_{i=1}^{F} E_i = \sum_{j=1}^{V} d_j = 2E. \tag{0.1}$$

Proof. The fact that $\sum_{i=1}^{F} E_i = 2E$ was already verified in the proof of Theorem 5. The identity $\sum_{j=1}^{V} d_j = 2E$ follows as the corresponding identity for graphs, as for polytopes it is also true that every edge contributes with 2 to the addition of the degrees of all the vertices. □

Corollary 7. *For any polytope P, the inequalities $3F \leq 2E$ and $3V \leq 2E$ hold.*

Proof. We just need to apply Remark 2 and Proposition 6. □

Now that we have lower bounds for the number of edges, let us find upper bounds for the same.

Proposition 8. *Let P be a polytope. Then $E \leq 3F - 6$ and $E \leq 3V - 6$.*

Proof. Since it follows from the previous corollary that $3V \leq 2E$, using Euler's formula for polytopes, we obtain that

$$E = V + F - 2 \leq \frac{2}{3}E + F - 2,$$

which implies that $\frac{E}{3} \leq F - 2$ or, equivalently, $E \leq 3F - 6$. We can proceed similarly to argue that $E \leq 3V - 6$. □

Proposition 9. *For a polytope P, the following statements hold.*

(1) *P has a face whose boundary consists of at most 5 edges.*
(2) *P has a vertex with degree at most 5.*

Proof. (1) Suppose, by way of contradiction, that $E_i \geq 6$ for every $i \in [F]$. Since

$$2E = \sum_{i=1}^{F} E_i \geq 6F,$$

it follows from the previous proposition that $3F - 6 \geq E \geq 3F$, which is a contradiction.

(2) The argument for this part is similar to that given in part (1). □

Let us say that a polytope is *fully symmetric* if all its faces equal to the same regular polygon and all its vertices have the same degree. We are in a position to determine all fully symmetric polytopes.

Example 10. Let P be a fully symmetric polytope whose vertices have degree d and whose faces are regular polygons whose boundaries consist of ℓ edges. It follows from Remark 2 and Proposition 9 that $3 \le d, \ell \le 5$. We split the rest of our consideration into three cases.

Case 1: $d = 3$. Then $3V = \sum_{j=1}^{V} d = 2E$, and it follows from the Euler's formula for polytopes that

$$2E = 3V = 3(E - F + 2) = 3E - 3F + 6,$$

and so $E = 3F - 6$. This implies that $\ell F = \sum_{i=1}^{F} E_i = 2E = 6F - 12$. Hence F divides 12. Since no polytope can have 3 faces (why?), we see that $F \in \{4, 6, 12\}$.

- If $F = 4$, then $E = 3F - 6 = 6$, and so $V = E - F + 2 = 4$. There exists indeed a fully symmetric polytope with 4 faces, 6 edges, and 4 vertices: the tetrahedron.
- If $F = 6$, then $E = 3F - 6 = 12$, and so $V = E - F + 2 = 8$. In this case, we obtain the cube.
- If $F = 12$, then $E = 3F - 6 = 30$, and so $V = E - F + 2 = 20$. This is a polytope called the *dodecahedron*.

Case 2: $d = 4$. Then $2E = \sum_{j=1}^{V} d = 4V$, and so

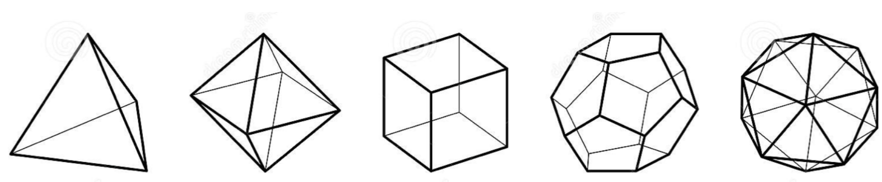
$$E = 2V = 2(E - F + 2) = 2E - 2F + 4,$$

which implies that $E = 2F - 4$. Therefore $\ell F = \sum_{i=1}^{F} E_i = 2E = 4F - 8$, that is, $(4 - \ell)F = 8$. Since $3 \le \ell \le 5$, this implies that $\ell = 3$ and $F = 8$. As a result, $E = 2F - 4 = 12$, and so $V = E - F + 2 = 6$. The fully symmetric polytope with 8 faces, 12 edges, and 6 vertices is called the *octahedron*. It can be obtained by gluing two idential Egyptian pyramids with their (square) bases.

Case 3: $d = 5$. Then $2E = \sum_{j=1}^{V} d = 5V$, and so

$$2E = 5V = 5(E - F + 2) = 5E - 5F + 10,$$

which implies that $3E = 5F - 10$. Therefore $3\ell F = 3\sum_{i=1}^{F} E_i = 6E = 10F - 20$, that is, $(10 - 3\ell)F = 20$. Since $3 \le \ell \le 5$, this implies that $\ell = 3$ and $F = 20$. As a result, $E = \frac{1}{3}(5F - 10) = 30$, and so $V = E - F + 2 = 12$. The fully symmetric polytope with 20 faces, 30 edges, and 12 vertices is called the *icosahedron*. Therefore there are exactly five fully symmetric polytopes, which are the so-called *Platonic solids*:



Practice Exercises

Exercise 1. *See Practice Midterm 4.*

Exercise 2. *See Practice Midterm 4.*

References

[1] M. Bóna: *A Walk Through Combinatorics: An Introduction to Enumeration and Graph Theory* (Fourth Edition), World Scientific, New Jersey, 2017.

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