MIT 18.211: COMBINATORIAL ANALYSIS

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Lecture 33: Euler's and Kuratowski's Theorems

In this lecture, we discuss graphs that can be drawn in the plane in such a way that no two edges cross each other. We state and prove a necessary condition for a graph to have this property (the Euler's formula), and finally we state (without proof) a characterization of these graphs (the Kuratowski's theorem).

Definition 1. A graph G is called *planar* if there is a way to draw G in the plane so that no two distinct edges of G cross each other.

Let G be a planar graph (not necessarily simple). Suppose that G is drawn in the plane so that no two of its edges cross each other. Then G subdivides the plane into regions, one of them unbounded and the rest of them bounded and containing no point of G inside. Such regions are called *faces* of G. As we prove in the next theorem, the number of faces of G does not depend on how we draw G in the plane. Throughout this lecture, we let F denote the number of faces of G. For the sake of consistency and to simplify notation, we set $V := |V(G)|$ and $E := |E(G)|$.

In the proof of our next theorem, we need the following definition. An edge e of a graph is called a *bridge* if the graph obtained from G by removing e has more connected component than G.

Theorem 2 (Euler's Theorem)**.** *Let G be a connected planar graph. Then the identity $V - E + F = 2$ holds.*

Proof. We proceed by induction on E. If $E = 1$, then the fact that G is connected guarantees that either G has one vertex v and a loop at v (i.e., an edge connecting v to itself) or G has to vertices and an edge connecting these two vertices. In the first case, $V = 1$, $E = 1$, and $F = 2$ (the interior and the exterior of the loop), and so $V - E + F = 1 - 1 + 2 = 2$. In the second case, $V = 2$, $E = 1$, and $F = 1$ (the unbounded face), and so $V - E + F = 2 - 1 + 1 = 2$.

Now suppose that the statement of the theorem holds for any connected planar graph with less than m edges for some $m \in \mathbb{N}_{\geq 2}$, and suppose that G is a connected planar graph with $E = m$. We split the rest of the proof into two cases.

Case 1: Every edge of G is a bridge. In this case, G has no loops and no multiple edges between any two vertices, and so G is a simple connected graph. As a consequence, G is a simple graph that is minimally connected or, equivalently, a tree. Since G is a tree, we see that $V = E + 1$ and $F = 1$ (the only face is the unbounded face). Hence $V - E + F = (E + 1) - E + 1 = 2$.

Case 2: G has no bridges. Let e be an edge of G (recall that $E \geq 2$). Let f_1 and f_2 be the faces in both sides of e. Observe that $f_1 \neq f_2$ as, otherwise, e would be a bridge. Let G' be the graph we obtain after removing e from G, and let V', E', and F' denote the number of vertices, edges, and faces of G', respectively. Clearly, $V = V'$ and $E = E' + 1$. In addition, $F = F' + 1$ because, after we delete e, the faces f_1 and f_2 are merged into one face and no other face of G is affected. In addition, G' is connected because e is not a bridge of G. Since $E' < E$, it follows from our induction hypothesis that $V' - E' + F' = 2$. Therefore

$$V - E + F = V' - (E' + 1) + (F' + 1) = V' - E' + F' = 2.$$

□

The previous theorem can be used to show that certain graphs are not planar. Let us take a look at two important small graphs that are not planar.

Example 3. Let us show that the complete graph K_5 is not planar. Suppose, by way of contradiction, that K_5 is planar. Then it follows from Euler's theorem that $V - E + F = 2$. We certainly know that $V = 5$ and $E = \binom{5}{2} = 10$. Therefore $F = E - V + 2 = 10 - 5 + 2 = 7$. Observe that we can draw G in the plane without crossing edges so that the boundary of its unbounded face consists of three edges, in which case, the boundary of any face of K_5 must consist of three edges (this is because any two distinct vertices of K_5 are adjacent). Hence we can count the edges of K_5 by counting the edges of each face and then dividing the obtained number by 2 to compensate for the fact that each edge is in the boundary of exactly two faces: doing so, we obtain that $E = \frac{3 \cdot 7}{2} = 11.5$, a contradiction. Thus, K_5 is not planar.

Example 4. We will argue now that the complete bipartite graph $K_{3,3}$ is not planar. As before, we assume, towards a contradiction, that $K_{3,3}$ is planar and use Euler's theorem to obtain that $F = E - V + 2 = 9 - 6 + 2 = 5$. Since $K_{3,3}$ is a bipartite graph, it has no cycles of length 3, and so the boundary of each face of $K_{3,3}$ consists of 4 edges. Thus, the number of edges of $K_{3,3}$ can be obtained by counting the edges of each of the 5 faces of $K_{3,3}$ and then dividing by 2 to compensate for double counting: doing so, we obtain that $9 = E = \frac{4 \cdot 5}{2} = 10$, a contradiction.

We conclude with an important theorem about planarity that characterizes planar graphs and is due to Kuratowski. First, we need a couple of definitions. For a graph G, let e be an edge of G and let v be a vertex of G of degree 2. Consider the following transformations of G.

(1) We add a new vertex to G in the middle of the edge e.
(2) We replace v and both edges incident to v by a new edge connecting the neighbors of v (possibly only one neighbor).

A graph G' is called *edge-equivalent* to G provided that G' can be obtained from G by a sequence of finitely many transformation as those described above.

Theorem 5 (Kuratowski's Theorem)**.** *A graph is non-planar if and only if it contains a subgraph that is edge-equivalent to either K_5 or $K_{3,3}$.*

Practice Exercises

Exercise 1. *Show that the Petersen graph is not planar.*

Exercise 2. *Let G be a simple planar graph. Prove that G has a vertex of degree at most 5.*

References

[1] M. Bóna: *A Walk Through Combinatorics: An Introduction to Enumeration and Graph Theory* (Fourth Edition), World Scientific, New Jersey, 2017.

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