MIT 18.211: COMBINATORIAL ANALYSIS

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Lecture 32: Brooks' Theorem

For a simple graph G, we let $\Delta(G)$ denote the maximum of all degrees of the vertices of G, that is, $\Delta(G) = \max\{\deg v \mid v \in V(G)\}$. A simple graph G is called *k-regular*, if any two vertices of G have the same degree, that is, $\deg v = \Delta(G)$ for every $v \in V(G)$.

Example 1. A path graph P is regular if and only if its length is 1, in which case $\Delta(P) = 1$. If a path graph P has length at least two, then it is not regular as it contains vertices of degree 1 and at least one vertex of degree 2; in this case, $\Delta(P) = 2$.

Example 2. For every $n \geq 3$, the graph C_n is 2-regular, and so $\Delta(G) = 2$.

Example 3. The complete graph K_n is the $(n-1)$-regular graph with n vertices. In this case, $\Delta(K_n) = n - 1$.

The main purpose of this lecture is to prove Brooks' Theorem, which gives an upper bound for the chromatic number of simple connected graphs with two exceptions. For the proof we present here, we need the following definition: in a simple graph G, a *cut-vertex* $v \in V(G)$ is a vertex satisfying that $G \setminus \{v\}$ has more connected components than G does.

Observe that $\chi(C_{2n+1}) = 3 = \Delta(C_{2n+1}) + 1$ and $\chi(K_n) = n = \Delta(K_n) + 1$ for every $n \in \mathbb{N}$. As the following theorem indicates, odd-length cycles and complete graphs are the only simple connected graphs G satisfying $\chi(G) > \Delta(G)$.

Theorem 4 (Brooks' Theorem)**.** *Let G be a simple connected graph. If G is neither complete nor an odd cycle, then $\chi(G) \leq \Delta(G)$.*

Proof. Set $k := \Delta(G)$. Since G is not complete, $k \geq 2$. If $k = 2$, the fact that G is connected implies that G is a cycle, and so G must be an even cycle. As every even cycle is bipartite, we see that $\chi(G) = 2 = k$ in this case. Assume from now on that $k \geq 3$, and fix k colors. We split the rest of the proof in the following cases.

Case 1: The graph G is not k-regular. In this case, there is a vertex $v_n \in V(G)$ with $\deg v_n \leq k - 1$. Since G is connected we can take a spanning tree T of G and label the vertices of G by $v_n, v_{n-1}, \dots, v_2, v_1$ coloring first the vertices which are closer to v_n in the tree T (that is, we label the vertices decreasingly as we travel the v_n-rooted tree T by levels). Thus, for any $j \in [n-1]$, the vertex v_j has at most $k-1$ adjacent vertices to its left in the sequence $v_1, v_2, \dots, v_n$; this is because v_j has at least one adjacent vertex in the subsequence $v_{j+1}, \dots, v_n$ (namely, the first vertex in its path to v_n through T). Observe that v_n also have at most $k-1$ adjacent to its left in the sequence $v_1, \dots, v_{n-1}$ because $\deg v_n \le k-1$. Then we can properly k-color G as follows: color v_1 using one of the k colors, and once we have properly k-colored $v_1, \dots, v_j$, if still $j < n$, then color v_{j+1} with a color that is different from the colors of its $k-1$ adjacent vertices in $v_1, \dots, v_j$. Hence we have found a proper k-color of G, and so $\chi(G) \le k$ in this case.

Case 2: The graph G is k-regular. We will again split the rest of the proof into two parts.

Case 2.1: The graph G contains a cut-vertex. Assume that one of the k fixed colors is red. Let v be a cut-vertex of G, and let $G_1, \dots, G_m$ be the connected components we obtain after removing v from G. Since v is a cut-vertex, $m \ge 2$. Fix $j \in [m]$, and consider the subgraph G'_j of G induced by the vertices $V(G) \cup \{v\}$. Since $m \ge 1$, the vertex v is adjacent to some vertex in $V(G) \setminus V(G'_j)$, and so $\deg_{G'_j} v \le k-1$. Then we can take a spanning tree of G'_j rooted at v as we did in Case 1, and use this tree to obtain a proper k-coloring of G'_j. After a possible renaming of colors, we can assume that in the obtained proper k-coloring of G'_j, the vertex v is red. Thus, we have obtained, for each $j \in [m]$, a proper k-coloring of G'_j (all of such coloring using the same k prescribed colors), and in each of such coloring, v is red. Since there are not edges from G_i to G_j if $i \ne j$ (because v is a cut-vertex of G), the proper k-coloring we have found for $G'_1, \dots, G'_m$ give rise to a proper k-coloring of G. Thus, $\chi(G) \le k$ also holds in this case.

Case 2.2: The graph G does not contain any cut-vertex. Fix $v \in V(G)$, and consider the graph $G' := G \setminus \{v\}$. Since G has no cut-vertices, G' is a connected graph. We split the rest of the proof into the following two cases.

Case 2.2.1: The graph G' contains no vertex-cut. Set $v_1 := x$. Since G is k-regular but not complete, v_1 cannot be adjacent to the rest of the vertices. This, along with the fact that G', ensures the existence of $v_2 \in V(G)$ such that there is path of length 2 in G from v_1 to v_2, and so there is a vertex v_n that is adjacent to both v_1 and v_2. Since v_2 is not a cut-vertex of G', we can take a spanning tree T of the graph $G \setminus \{v_1, v_2\}$ rooted at v_n and, as in Case 1, we can label the vertices $G \setminus \{v_1, v_2\}$ using the tree T so that for every $j \in [\![3, n-1]\!]$ the vertex v_j has at most $k-1$ adjacent to its left in the sequence $v_1, v_2, v_3, \dots, v_{n-1}, v_n$. Then we can properly color v_1 and v_2 with the same color (as they are not adjacent), and then we can properly color the vertices $v_3, \dots, v_{n-1}$ as we did in Case 1. Now observe that although v_n has k adjacent vertices, two of them share the same color, and so we can choose one of the k prescribed colors for v_n to obtain a proper vertex k-coloring for G.

Case 2.2.2: This part will be an exercise in Problem Set 6. $\square$

Practice Exercises

Exercise 1. *Show that a complete graph has no cut-vertex.*

Exercise 2. *Which are the cut vertex of a forest?*

References

[1] M. Bóna: *A Walk Through Combinatorics: An Introduction to Enumeration and Graph Theory* (Fourth Edition), World Scientific, New Jersey, 2017.

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