MIT 18.211: COMBINATORIAL ANALYSIS

FELIX GOTTI

Lecture 31: Chromatic Numbers and Polynomials

Chromatic Numbers. For $k \in \mathbb{N}$, a *proper k-coloring* of a simple graph G is a (coloring) function $f : V(G) \to [k]$ such that no two adjacent vertices of G have the same image under f. We can think of every index in the codomain of f as the label of a color we can use to color some of the vertices of G via f. We say that a simple graph G is *k-colorable* if it admits a proper k-coloring. It is clear that if G is k-colorable, then it is m-colorable for every $m \geq k$.

Definition 1. The *chromatic number* of a simple graph G, denoted by $\chi(G)$, is the minimum among all $k \in \mathbb{N}$ such that G is k-colorable.

Let us take a look at a few examples.

Example 2. For $n \in \mathbb{N}$, let G be a graph satisfying $|V(G)| = n$ and $|E(G)| = 0$. Since G does not have any adjacency relation, we easily see that $\chi(G) = 1$. In this case, a proper 1-coloring is given by the constant function $f : V(G) \to \{1\}$.

Example 3. Suppose that G is a bipartite graph on the parts X and Y. If $E(G)$ is empty, then G is one of the graphs in Example 2, and so $\chi(G) = 0$. Otherwise, $\chi(G) \geq 2$. Verifying that $\chi(G) = 2$ amounts to observing that we can color the vertices in X black and the vertices in Y white, which yield the proper 2-coloring $f : V(G) \to \{1, 2\}$ defined by $f(v) = 1$ if and only if $v \in X$. On the other hand,it is clear that every 2-colorable graph is bipartite graph on the parts determine by the two colors. This example explains why often the parts of a bipartite graph are called coloring/color classes. In particular, we obtain that

- every graph consisting of a path is 2-colorable,
- every tree is 2-colorable, and
- the complete bipartite graphs $K_{m,n}$ are 2-colorable.

Example 4. Let us find the chromatic number of C_n for each $n \in \mathbb{N}$ with $n \geq 3$. If n is even, then C_n is bipartite, and so $\chi(C_n) = 2$. Suppose then that n is odd. Since C_n is not bipartite, $\chi(C_n) \geq 3$. Now write $n = 2k + 1$ for some $k \in \mathbb{N}$, and set $V(C_n) = \{v_1, v_2, \dots, v_{2k+1}\}$ such that $v_1 v_2 \dots v_{2k+1} v_1$ is the only cycle of C_n. To exhibit a proper 3-coloring of G, color the vertices in $\{v_{2j-1} \mid j \in [k]\}$ blue, the vertices in $\{v_{2j} \mid j \in [k]\}$ green, and the vertex v_{2k+1} red. Hence $\chi(C_n) = 3$ when n is odd.

Every graph G we have seen so far admits a proper 3-coloring, regardless of the sizes of $V(G)$ and $E(G)$. For certain graphs (with large numbers of vertices and edges), indeed, just a few colors suffices to obtain proper coloring. For instance, the famous 4-Color Theorem, which was an open conjecture for quite a while, states that 4 colors suffices to color the regions of any map in such a way that no two regions sharing a border have the same color. The graph modeling the regions of a man and the adjacency/border relations between regions has the property of being *planar*, and we will discuss them in coming lectures. For now, let us finish this subsection illustrating with a very simple example that there are graphs whose vertex colorings have at least as many colors as their numbers of vertices.

Example 5. For $n \in \mathbb{N}$, let K_n be the complete graph on $[n]$. Since any two distinct vertices of K_n are adjacent, in order to have a proper coloring of K_n not two vertex can have the same color. From this observation, it follows immediately that $\chi(K_n) = n$.

Chromatic Polynomials. In this subsection we introduce an important tool to study graph coloring, the chromatic polynomial.

Proposition 6. *Let G be a simple graph with labeled vertices. For every $k \in \mathbb{N}$, let p_k denote the number of proper k-coloring of G. Then there exists a unique polynomial $p(x) \in \mathbb{Q}[x]$ such that $p(k) = p_k$ for every $k \in \mathbb{N}$.*

Proof. Set $n := |V(G)|$. For each $m \in \mathbb{N}$, let c_m denote the number of proper m-coloring of G that use all the m colors. It is clear that $c_m = 0$ when $m > n$. Observe that, for each $k \in \mathbb{N}$, the set of proper k-coloring of G can be counted as follows: for each $m \in [\![1, n]\!]$, we choose in $\binom{k}{m}$ ways the m colors that we will actually use in the proper k-coloring of G (here $\binom{k}{m} = 0$ if $m > k$), and then we properly color the vertices of G using *all* the m chosen colors in c_m possible ways. As a result, we obtain that

$$p_k = \sum_{m=1}^{n} \binom{k}{m} c_m. \tag{0.1}$$

With the identity (0.1) in mind, we now consider the polynomial $p(x) := \sum_{m=1}^{n} \binom{x}{m} c_m$, where $\binom{x}{m} = \frac{x(x-1)\cdots(x-m+1)}{m!}$. It is clear that $p(x)$ is a polynomial with coefficients in $\mathbb{Q}$. In addition, any other polynomial $q(x) \in \mathbb{Q}[x]$ such that $q(k) = p_k$ will be equal to $p(x)$ because they coincide at infinitely many real numbers, namely, $\mathbb{N}$. $\square$

We are in a position now to define the chromatic polynomial of a simple graph.

Definition 7. With notation as in Proposition 6, the polynomial $p(x)$ is called the *chromatic polynomial* of the graph G.

Example 8. For $n \in \mathbb{N}$, let G be the graph on $[n]$ with $|E(G)| = 0$. Since there is no adjacency relations in G, given k colors, we can choose any of the k colors for each vertex to obtain a proper k-coloring. So there is a total of k^n proper k-colorings of G. Hence the chromatic polynomial of G is $p(x) = x^n$.

Example 9. For $n \in \mathbb{N}$, let P_n be the path graph on $[n]$ with adjacency relations ij for any $i \in [n-1]$ and $j = i+1$. In order to properly color $V(P_n)$ using k color, we can pick any of the k colors for the vertex 1, any of the $k-1$ colors different from that of vertex 1 to color vertex 2, and so on until we have to pick any of the $k-1$ colors different from that of vertex $n-1$ to color vertex n. Thus, the chromatic polynomial of P_n is $p(x) = x(x-1)^{n-1}$.

Example 10. For $n \in \mathbb{N}$, let K_n be the complete graph on $[n]$. To properly color $V(K_n)$ with k color, k must be at least n, in which case, we can use k color for the vertex 1, then $k-1$ color for the vertex 2, and so on until we use $k-n+1$ colors for the vertex n. Therefore the chromatic polynomial of K_n is

$$p(x) = x(x-1)\cdots(x-n+1) = n!\binom{x}{n}.$$

Practice Exercises

Exercise 1. *Find the chromatic polynomial of a tree with n vertices.*

Exercise 2. *For $n_1, n_2, n_3 \in \mathbb{N}$, find the chromatic number and the chromatic polynomial of the complete tripartite graph K_{n_1,n_2,n_3} on the parts $[n_1]$, $[n_1+n_2] \setminus [n_1]$, and $[n_1+n_2+n_3] \setminus [n_1+n_2]$.*

References

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Department of Mathematics, MIT, Cambridge, MA 02139
Email address: fgotti@mit.edu