Hall’s Theorem. Let $G$ be a simple graph, and let $S$ be a subset of $E(G)$. If no two edges in $S$ form a path, then we say that $S$ is a matching of $G$. A matching $S$ of $G$ is called a perfect matching if every vertex of $G$ is covered by an edge of $S$.

Definition 1. Let $G$ be a bipartite graph on the parts $X$ and $Y$, and let $S$ be a matching of $G$. If every vertex in $X$ is covered by an edge of $S$, then we say that $S$ is a perfect matching of $X$ into $Y$.

For a graph $G$ and a subset $T$ of $V(G)$, we let $N_G(T)$ denote the set of vertices of $G$ that are adjacent to some vertex in $T$, that is,

$$N_G(T) := \{v \in V(G) \mid vw \in E(G) \text{ for some } w \in T\}.$$ 

Observe that if $G$ is bipartite on the parts $A$ and $B$, then $N_G(T) \subseteq B$ for any $T \subseteq A$.

We proceed to prove the main result of this lecture, which is due to Philip Hall and is often called Hall’s Marriage Theorem.

Theorem 2. For a bipartite graph $G$ on the parts $X$ and $Y$, the following conditions are equivalent.

(a) There is a perfect matching of $X$ into $Y$.

(b) For each $T \subseteq X$, the inequality $|T| \leq |N_G(T)|$ holds.

Proof. (a) $\Rightarrow$ (b): Let $S$ be a perfect matching of $X$ into $Y$. As $S$ is a perfect matching, for every $x \in X$ there exists a unique $y_x \in Y$ such that $xy_x \in S$. Define the map $f: X \rightarrow Y$ by $f(x) = y_x$. Since $S$ is a matching, the function $f$ is injective. Therefore for any $T \subseteq X$, we see that $|T| = |f(T)| \leq |N_G(T)|$ because $f(T) \subseteq N_G(T)$.

(b) $\Rightarrow$ (a): Conversely, suppose that $|T| \leq |N_G(T)|$ for each $T \subseteq X$. We will prove that there exists a perfect matching of $X$ into $Y$ by induction on $n := |X|$. If $n = 1$, then the only vertex $x$ in $X$ must be adjacent to some vertex $y$ in $Y$ by condition (b) and, therefore, $\{xy\}$ is a perfect matching of $X$ into $Y$. Now assume that every bipartite graph on the parts $X'$ and $Y'$ with $|X'| < |X|$ and satisfying condition (b) has a perfect matching of $X'$ into $Y'$. We split the rest of the proof into two cases.

Case 1: For every nonempty proper subset $T$ of $X$ (that is, $T \subsetneq X$), the strict inequality $|T| < |N_G(T)|$ holds. Take $x \in X$ and $y \in N_G(\{x\})$. Let $G'$ be the bipartite graph we
obtain by removing \(x\) and \(y\) (and the edges incident to them) from \(G\).

Now for every subset \(A\) of \(X \setminus \{x\}\), we see that
\[
|N_G'(A)| \geq |N_G(A)| - 1 \geq |A|,
\]
where the last inequality holds because \(A\) is a strict subset of \(X\). By induction hypothesis, there exists a perfect matching \(S'\) in \(G'\) of \(X \setminus \{x\}\) into \(Y \setminus \{y\}\). It is clear now that \(S' \cup \{xy\}\) is a perfect matching in \(G\) of \(X\) into \(Y\).

**Case 2:** There exists a nonempty proper subset \(A\) of \(X\) such that \(|A| = |N_G(A)|\). Let \(G_1\) be the subgraph of \(G\) induced by the set of vertices \(A \cup N_G(A)\), and let \(G_2\) be the subgraph of \(G\) we obtain by removing \(A \cup N_G(A)\) (and their incident edges) from \(G\).

It is clear that \(G_1 = (A, N_G(A))\) and \(G_2 = (X \setminus A, Y \setminus N_G(A))\) are bipartite graphs. Let us show that both \(G_1\) and \(G_2\) satisfy condition (b).

To show that \(G_1\) satisfies (b), take \(T \subseteq A\). It follows by the way \(G_1\) was constructed that \(N_{G_1}(T) = N_G(T)\). As a result, \(|N_{G_1}(T)| = |N_G(T)| \geq |T|\). Then \(G_1\) satisfies condition (b). In order to argue that \(G_2\) also satisfies condition (b), take \(T' \subseteq X \setminus A\) and observe that \(N_G(T' \cup A) = N_G(A) \cup N_{G_1}(T')\), where the union on the right-hand side is disjoint. Since \(|N_G(T' \cup A)| \geq |T' \cup A|\) and \(|N_G(A)| = |A|\),
\[
|N_{G_2}(T')| = |N_G(T' \cup A)| - |N_G(A)| \geq |T' \cup A| - |A| = (|T'| + |A|) - |A| = |T'|.
\]
Therefore \(G_2\) also satisfies condition (b). Since \(|A| < |X|\) and \(|X \setminus A| < |X|\), our induction hypothesis guarantees the existence of a perfect matching \(S_1\) in \(G_1\) of \(A\) into \(N_G(A)\) and a perfect matching \(S_2\) in \(G_2\) of \(X \setminus A\) into \(Y \setminus N_G(A)\). Then it follows from the construction of \(G_1\) and \(G_2\) that \(S_1 \cup S_2\) is a perfect matching in \(G\) of \(X\) into \(Y\), which concludes the proof. \(\square\)

We conclude this lecture characterizing whether a matching on a simple graph has the maximum number of edges possible. First, we need the following definitions.

**Definition 3.** Let \(G\) be a graph, and let \(M\) be a matching of \(G\). A path \(P = v_1v_2 \ldots v_k\) is called \(M\)-alternating provided that \(v_{i-1}v_i \in M\) if and only if \(v_iv_{i+1} \notin M\). An \(M\)-alternating path is called \(M\)-augmenting if it starts and ends at vertices that are not covered by any edge of \(M\).

We can now characterize the maximum-length matching in terms of augmenting paths.

**Theorem 4.** Let \(G\) be a simple graph with a matching \(M\). Then \(M\) is a maximum-length matching if and only if \(G\) has no \(M\)-augmenting paths.

**Proof.** For the direct implication suppose that \(G\) has an \(M\)-augmenting path, namely, \(P\). Since \(P\) is an \(M\)-augmenting path, it has odd length. Write \(P := v_1v_2 \ldots v_{2k}\). Since \(P\) is \(M\)-alternating and none of the vertices \(v_1\) and \(v_{2k}\) is covered by \(M\), we see that \(P \cap M = \{v_{2k}v_{2k+1} \mid k \in [\ell - 1]\}\). Then after replacing the subset \(P \cap M\) of \(M\) by
the subset \( \{v_{2k-1}v_{2k} \mid k \in [\ell]\} \), we would obtain a new matching of \( G \) with more edges than \( M \).

For the reverse implication, suppose that \( G \) has no \( M \)-augmenting path. Let \( M' \) be a maximum-length matching. If \( M' = M \), then we are done. Suppose, therefore, that \( M \neq M' \). Let \( S \) be the symmetric difference of \( M \) and \( M' \), that is, the set of edges in \( M \cup M' \) that are not in \( M \cap M' \). Now observe that every connected component of the graph \((V(G), S)\) is either a path or an (even-length) cycle whose edges alternate between \( M' \) and \( M \). Now the maximality of \( M' \), along with the non-existence of \( M \)-augmenting paths, guarantees that each connected component of \((V(G), S)\) that is a path must be a path of even length. Hence \( |M| = |M'| \), which implies that \( M \) is a maximum-length cycle. \( \square \)

**Practice Exercises**

**Exercise 1.** [1, Exercise 11.4] Let \( G \) be a bipartite graph on the parts \( X \) and \( Y \), and suppose that the inequality \( \deg x \geq \deg y \) holds for all \( x \in X \) and \( y \in Y \). Prove that \( X \) has a perfect matching into \( Y \).

**Exercise 2.** [1, Exercise 11.12] Let \( G \) be a regular bipartite graph (that is, a graph with all the vertices having the same degree). Prove that \( G \) has a perfect matching.

**References**


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