MIT: 18.211: COMBINATORIAL ANALYSIS

FELIX GOTTI

Lecture 30: Matching and Hall's Theorem

Hall's Theorem. Let G be a simple graph, and let S be a subset of $E(G)$. If no two edges in S form a path, then we say that S is a *matching* of G. A matching S of G is called a *perfect matching* if every vertex of G is covered by an edge of S.

Definition 1. Let G be a bipartite graph on the parts X and Y, and let S be a matching of G. If every vertex in X is covered by an edge of S, then we say that S is a *perfect matching* of X *into* Y.

For a graph G and a subset T of $V(G)$, we let $N_G(T)$ denote the set of vertices of G that are adjacent to some vertex in T, that is,

$$N_G(T) := \{v \in V(G) \mid vw \in E(G) \text{ for some } w \in T\}.$$

Observe that if G is bipartite on the parts A and B, then $N_G(T) \subseteq B$ for any $T \subseteq A$. We proceed to prove the main result of this lecture, which is due to Philip Hall and is often called Hall's Marriage Theorem.

Theorem 2. *For a bipartite graph G on the parts X and Y, the following conditions are equivalent.*

(a) *There is a perfect matching of X into Y.*
(b) *For each $T \subseteq X$, the inequality $|T| \le |N_G(T)|$ holds.*

Proof. (a) $\Rightarrow$ (b): Let S be a perfect matching of X into Y. As S is a perfect matching, for every $x \in X$ there exists a unique $y_x \in Y$ such that $xy_x \in S$. Define the map $f\colon X \to Y$ by $f(x) = y_x$. Since S is a matching, the function f is injective. Therefore for any $T \subseteq X$, we see that $|T| = |f(T)| \le |N_G(T)|$ because $f(T) \subseteq N_G(T)$.

(b) $\Rightarrow$ (a): Conversely, suppose that $|T| \le |N_G(T)|$ for each $T \subseteq X$. We will prove that there exists a perfect matching of X into Y by induction on $n := |X|$. If $n = 1$, then the only vertex x in X must be adjacent to some vertex y in Y by condition (b) and, therefore, $\{xy\}$ is a perfect matching of X into Y. Now assume that every bipartite graph on the parts X' and Y' with $|X'| < |X|$ and satisfying condition (b) has a perfect matching of X' into Y'. We split the rest of the proof into two cases.

Case 1: For every nonempty proper subset T of X (that is, $T \subsetneq X$), the strict inequality $|T| < |N_G(T)|$ holds. Take $x \in X$ and $y \in N_G(\{x\})$. Let G' be the bipartite graph we obtain by removing x and y (and the edges incident to them) from G. Now for every subset A of $X \setminus \{x\}$, we see that

$$|N_{G'}(A)| \geq |N_G(A)| - 1 \geq |A|,$$

where the last inequality holds because A is a strict subset of X. By induction hypothesis, there exists a perfect matching S' in G' of $X \setminus \{x\}$ into $Y \setminus \{y\}$. It is clear now that $S' \cup \{xy\}$ is a perfect matching in G of X into Y.

Case 2: There exists a nonempty proper subset A of X such that $|A| = |N_G(A)|$. Let G_1 be the subgraph of G induced by the set of vertices $A \cup N_G(A)$, and let G_2 be the subgraph of G we obtain by removing $A \cup N_G(A)$ (and their incident edges) from G. It is clear that $G_1 = (A, N_G(A))$ and $G_2 = (X \setminus A, Y \setminus N_G(A))$ are bipartite graphs. Let us show that both G_1 and G_2 satisfy condition (b).

To show that G_1 satisfies (b), take $T \subseteq A$. It follows by the way G_1 was constructed that $N_{G_1}(T) = N_G(T)$. As a result, $|N_{G_1}(T)| = |N_G(T)| \geq |T|$. Then G_1 satisfies condition (b). In order to argue that G_2 also satisfies condition (b), take $T' \subseteq X \setminus A$ and observe that $N_G(T' \cup A) = N_G(A) \cup N_{G_2}(T')$, where the union on the right-hand side is disjoint. Since $|N_G(T' \cup A)| \geq |T' \cup A|$ and $|N_G(A)| = |A|$,

$$|N_{G_2}(T')| = |N_G(T' \cup A)| - |N_G(A)| \geq |T' \cup A| - |A| = (|T'| + |A|) - |A| = |T'|.$$

Therefore G_2 also satisfies condition (b). Since $|A| < |X|$ and $|X \setminus A| < |X|$, our induction hypothesis guarantees the existence of a perfect matching S_1 in G_1 of A into $N_G(A)$ and a perfect matching S_2 in G_2 of $X \setminus A$ into $Y \setminus N_G(A)$. Then it follows from the construction of G_1 and G_2 that $S_1 \cup S_2$ is a perfect matching in G of X into Y, which concludes the proof. □

We conclude this lecture characterizing whether a matching on a simple graph has the maximum number of edges possible. First, we need the following definitions.

Definition 3. Let G be a graph, and let M be a matching of G. A path $P = v_1 v_2 \dots v_\ell$ is called *M-alternating* provided that $v_{i-1}v_i \in M$ if and only if $v_i v_{i+1} \notin M$. An M-alternating path is called *M-augmenting* if it starts and ends at vertices that are not covered by any edge of M.

We can now characterize the maximum-length matching in terms of augmenting paths.

Theorem 4. *Let G be a simple graph with a matching M. Then M is a maximum-length matching if and only if G has no M-augmenting paths.*

Proof. For the direct implication suppose that G has an M-augmenting path, namely, P. Since P is an M-augmenting path, it has odd length. Write $P := v_1 v_2 \dots v_{2\ell}$. Since P is M-alternating and none of the vertices v_1 and v_ℓ is covered by M, we see that $P \cap M = \{v_{2k} v_{2k+1} \mid k \in [\ell - 1]\}$. Then after replacing the subset $P \cap M$ of M by the subset $\{v_{2k-1}v_{2k} \mid k \in [\ell]\}$, we would obtain a new matching of G with more edges than M.

For the reverse implication, suppose that G has no M-augmenting path. Let M' be a maximum-length matching. If $M' = M$, then we are done. Suppose, therefore, that $M \neq M'$. Let S be the symmetric difference of M and M', that is, the set of edges in $M \cup M'$ that are not in $M \cap M'$. Now observe that every connected component of the graph $(V(G), S)$ is either a path or an (even-length) cycle whose edges alternate between M' and M. Now the maximality of M', along with the non-existence of M-augmenting paths, guarantees that each connected component of $(V(G), S)$ that is a path must be a path of even length. Hence $|M| = |M'|$, which implies that M is a maximum-length cycle. □

Practice Exercises

Exercise 1. [1, Exercise 11.4] *Let G be a bipartite graph on the parts X and Y, and suppose that the inequality $\deg x \geq \deg y$ holds for all $x \in X$ and $y \in Y$. Prove that X has a perfect matching into Y.*

Exercise 2. [1, Exercise 11.12] *Let G be a regular bipartite graph (that is, a graph with all the vertices having the same degree). Prove that G has a perfect matching.*

References

[1] M. Bóna: *A Walk Through Combinatorics: An Introduction to Enumeration and Graph Theory* (Fourth Edition), World Scientific, New Jersey, 2017.

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