The Adjacency Matrix. A helpful way to represent a graph $G$ is by using a matrix that encodes the adjacency relations of $G$. This matrix is called the adjacency matrix of $G$ and facilitates the use of algebraic tools to better understand graph theoretical aspects. In the first part of this lecture, we provide a couple of applications of the adjacency matrix representation.

Definition 1. Let $G$ be a multigraph with $V(G) = [n]$. Then the adjacency matrix $A$ of $G$ is defined as follows:

- if $G$ is undirected, then $A_{jk}$ is the number of edges between $j$ and $k$, and
- if $G$ is directed, then $A_{jk}$ is the number of edges from $j$ to $k$.

We observe that the adjacency matrix of any undirected multigraph is symmetric. However, this is not always the case for the adjacency matrix of a directed multigraph. As we proceed to show, adjacency matrices can be used to compute number of walks in a graph.

Proposition 2. Let $G$ be a graph (directed or undirected) on $[n]$ with adjacency matrix $A$. For any $j, k \in [n]$ and $\ell \in \mathbb{N}$, there are $A_{jk}^\ell$ walks of length $\ell$ from $j$ to $k$.

Proof. We proceed by induction on $\ell$. If $\ell = 1$, then $A_{jk}$ is the number of edges from $j$ to $k$, which is the number of walks of length 1 from $j$ to $k$. Assume the statement of the proposition holds for $\ell \in \mathbb{N}$, and fix two vertices $j$ and $k$ of $G$. Set $B = A^\ell$. By the induction hypothesis, for any $v \in V(G)$ there are $B_{ju}^\ell$ walks of length $\ell$ from $j$ to $v$ and by the definition of $A$ there are $A_{vk}$ walks of length 1 (that is, edges) from $v$ to $k$. As every walk of length $\ell + 1$ from $j$ to $k$ can be obtained by concatenating, for some $v \in [n]$, a walk of length $\ell$ from $j$ to $v$ and a walk of length 1 (an edge) from $v$ to $k$, the number of walks of length $\ell + 1$ from $j$ to $k$ is

$$
\sum_{v=1}^{n} B_{ju}^\ell A_{vk} = (BA)_{jk} = (A^\ell A)_{jk} = A_{jk}^{\ell+1},
$$

which concludes the proof.

In addition, we can use the adjacency matrix to check whether the corresponding graph is connected. The following proposition shows how to do this.
Proposition 3. Let $G$ be a simple graph on $[n]$, and let $A$ be the adjacency matrix of $G$. Then $G$ is connected if and only if all the entries of $(I_n + A)^{n-1}$ are positive.

Proof. We know that $G$ is connected if and only if any two distinct vertices $j$ and $k$ of $G$ are connected by a path (of length at most $n - 1$), which happens if and only if $A_{j,k}^\ell > 0$ for some $\ell \in [n - 1]$. Therefore the statement of the proposition follows from the following (Newton-Binomial) identity:

$$(I_n + A)^{n-1} = \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} A^\ell.$$ \hfill \Box$$

The Matrix-Tree Theorem. Our next goal is to introduce another important matrix related to a given directed graph $G$, the incidence matrix, and use it to provide a formula for the number of spanning trees of $G$. This formula, in turns, will allow us to prove the Matrix-Tree Theorem, which expresses the number of spanning trees of an (undirected graph) as a determinant of certain matrix.

Let $G$ be a directed graph. In this lecture, we say that the **underlying graph** of $G$ is the graph we obtain from $G$ by ignoring the orientation of the edges. A **spanning tree** of a directed graph $G$ is a subgraph $T$ such that the underlying graph of $T$ is a spanning tree of the underlying graph of $G$.

**Definition 4.** Let $G$ be a directed graph with $V(G) = \{v_1, \ldots, v_m\}$ and $E(G) = \{e_1, \ldots, e_n\}$. The **incidence matrix** of $G$ is the $m \times n$ matrix $A$ with $A_{ij} = 1$ (resp., $A_{ij} = -1$) if the edge $e_j$ starts (resp., ends) at $v_i$ and with $A_{ij} = 0$ if $e_j$ is not connected to $v_i$.

**Theorem 5.** Let $G$ be a connected directed graph (without loops), and let $A$ be the incidence matrix of $G$. If $A_0$ is the matrix obtained from $A$ by removing the last row, then $\det(A_0 A_0^T)$ is the number of spanning trees of $G$.

Proof. Since $G$ is connected, $m - 1 \leq n$. Let $B$ be an $(m - 1) \times (m - 1)$ submatrix of $A_0$, and let $G'$ be the subgraph of $G$ with $V(G') = V(G)$ and whose edges correspond to the columns of $B$. We claim that $G'$ is a spanning tree of $G$ if and only if $|\det B| = 1$ and that, otherwise, $\det B = 0$. We proceed by induction on $|V(G)| \in \mathbb{N}_{\geq 2}$.

If $|V(G)| = 2$, then $G'$ consists of exactly one edge and so it is a spanning tree of $G$, while $B \in \{(1), (-1)\}$ and so $|\det B| = 1$. Now assume that our claim holds for any directed graph with $m - 1$ vertices (with $m \geq 3$), and suppose that $V(G) = \{v_1, \ldots, v_m\}$ and $E(G) = \{e_1, \ldots, e_n\}$. Let $B$ and $G'$ be as we have described before. We split the rest of the proof into two cases.

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1Because of time-constrains, this proposition was not covered in class, but I have included here to provide further applications of the adjacency matrix representation of a graph.
Case 1: There exists \( i \in [n-1] \) such that \( \text{indeg}_{G'}v_i + \text{outdeg}_{G'}v_i = 1 \). This implies that there is exactly one nonzero entry (either 1 or \(-1\)) in the \( i \)-th row of \( B \). Let \( e_j \) be the edge connected to \( v_i \) in \( G' \). If we compute \( \det B \) expanding along the \( i \)-th row of \( B \), then we obtain that \( |\det B| = |\det B'| \), where \( B' \) is the submatrix of \( B \) obtained by eliminating the \( i \)-th row and the column corresponding to \( e_j \). It follows by the induction hypothesis that \( \det B' \) is either 1 or \(-1\) if and only if \( G' \setminus \{v_i\} \) is a spanning tree of \( G \). As \( \text{indeg}_{G'}v_i + \text{outdeg}_{G'}v_i = 1 \), this happens if and only if \( G' \) is a spanning tree of \( G \). Similarly, the induction hypothesis allows us to deduce that \( \det B = 0 \) if and only if \( G' \) is not a spanning tree of \( G \).

Case 2: \( \text{indeg}_{G'}v_i + \text{outdeg}_{G'}v_i \neq 1 \) for any \( i \in [n-1] \). Since \( |V(G)| = m \) and \( |E(G')| = m - 1 \), the graph \( G' \) must have a vertex \( v_j \) such that \( \text{indeg}_{G'}v_j + \text{outdeg}_{G'}v_j = 0 \). In particular, \( G' \) is not a spanning tree of \( G \) (as it is not even a tree itself). If \( j \leq m - 1 \), then \( B \) has a row full of zeros and so \( \det B = 0 \). Otherwise, \( j = n \) and, therefore, every column of \( B \) has exactly one entry \(-1\), one entry 1, and the rest of the entries are zeros. This last statement implies that the addition of all the row vectors of \( B \) is the zero vector. In particular, the rows of \( B \) are linearly dependent, which implies that \( \det B = 0 \).

Now we can use our already-proved claim in tandem with the Binet-Cauchy Formula to complete the proof. It follows from the Binet-Cauchy Formula that

\[
(0.1) \quad \det(A_0 A_0^T) = \sum (\det B)^2,
\]

where the sum runs over all \((m - 1) \times (m - 1)\) submatrices \( B \) of \( A_0 \). By our claim, the spanning trees of \( G \) correspond to submatrices \( B \) with \( \det B \in \{\pm 1\} \), and the determinant of the rest of the \((m - 1) \times (m - 1)\) submatrices of \( A_0 \) is zero. Hence the identity \((0.1)\) implies that \( G \) contains exactly \( \det(A_0 A_0^T) \) spanning trees. \( \square \)

We are in a position to prove that Matrix-Tree Theorem.

**Theorem 6.** Let \( U \) be a simple (undirected graph) with \( V(U) = \{v_1, \ldots, v_m\} \). Let \( L \) be the \((m - 1) \times (m - 1)\) matrix defined by

\[
L_{ij} = \begin{cases} 
\deg v_i & \text{if } i = j \\
-1 & \text{if } i \neq j \text{ and } v_iv_j \in E(U) \\
0 & \text{otherwise.}
\end{cases}
\]

for all \( i, j \in [m - 1] \). Then the number of spanning trees of \( U \) is \( \det L \).

**Proof.** Let \( G \) be the directed graph that we obtain from \( U \) by replacing each edge of \( U \) by two arrows, one in each direction. Let \( A \) be the incidence matrix of \( G \), and let \( A_0 \) be the matrix we obtain from \( A \) by removing the last row. We claim that \( A_0 A_0^T = 2L \). Set \( M = A_0 A_0^T \) and observe that

\[
M_{ij} = A_{i1}A_{j1} + A_{i2}A_{j2} + \cdots + A_{in}A_{jn},
\]
where $n = |E(G)|$. When $i = j$, the summand $A_ikA_ik$ contributes with 1 to $M_{ii}$ if and only if the edge of $G$ determined by the $k$-th column of $A$ is incident to $v_i$. Therefore $M_{ii} = \text{indeg}_G v_i + \text{outdeg}_G v_i = 2 \deg_U v_i = 2L_{ii}$. On the other hand, if $i \neq j$, then $A_ikA_jk$ contributes with $-1$ to $M_{ij}$ if and only if the edge corresponding to the $k$-th column of $A$ connects $v_i$ and $v_j$, which happens exactly for two indices $k$. Thus, $M_{ij} = -2 = 2L_{ij}$ if $v_iv_j \in E(U)$. Otherwise, there are no edges connecting $v_i$ and $v_j$ in $U$ (or in $G$) and so $M_{ij} = 0 = 2L_{ij}$. Hence $A_0A_0^T = M = 2L$, as claimed. Therefore
\[
2^{m-1} \det L = \det(2L) = \det(A_0A_0^T),
\]
which is, by virtue of Theorem 4, the number of spanning trees of $G$. Since each spanning tree of $U$ gives rise to exactly $2^{m-1}$ spanning trees of $G$, we conclude that $\det L$ is the number of spanning trees of $U$. \qed

**Practice Exercises**

**Exercise 1.** Let $G$ be a directed graph such that $|E(G)| = |V(G)| - 1$. If $\text{indeg} v + \text{outdeg} v \neq 1$ for all $v \in V(G)$, argue that $G$ has an isolated vertex.

**Exercise 2.** Use the Matrix-Tree Theorem to rediscover that the number of spanning trees of $K_n$ is $n^{n-2}$.

**References**