MIT 18.211: COMBINATORIAL ANALYSIS

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Lecture 24: Cayley's Theorem

Given a set V consisting of n vertices, one can easily argue that there are $2^{\binom{n}{2}}$ graphs on the V. Indeed, there are $\binom{n}{2}$ pairs of vertices and, in order to build a graph on V, we merely have to decide for each of these pair of vertices whether to connected with an edge or not. If instead of graphs, we want to count the set of trees on V, we can still do so but the argument is not as simple. In this lecture, we will prove Cayley's Theorem, which state that there are n^{n-2} trees on V. First, we provide a characterization of trees in terms of paths, which we will use in the proof of Cayley's theorem.

Here is another characterization of a tree in terms of paths.

Proposition 1. *A simple graph is a tree if and only if for any distinct two vertices there exists exactly one path connecting them.*

Proof. Let G be a simple graph.

For the direct implication, assume that G is a tree and suppose, towards a contradiction, that there are two distinct vertices x and y of G and two distinct paths $v_1 v_2 \dots v_m$ and $w_1 w_2 \dots w_n$ connecting x and y. Let i be the minimum index such that $v_{i+1} \neq w_{i+1}$. Now suppose that j is the minimum index such that $j > i$ and w_j appears in the path $v_1 v_2 \dots v_m$. Suppose that $w_j = v_k$. Then $v_i v_{i+1} \dots v_k w_{j-1} \dots w_{i+1}$ is a cycle in G, which contradicts that G is a tree.

To argue the converse, assume that for any two distinct vertices of G there exists exactly one path in G connecting them. Now suppose, by way of contradiction, that G is not a tree. Thus, G must have a cycle $v_1 v_2 \dots v_\ell v_1$ and, therefore, $v_1 v_2$ and $v_1 v_\ell v_{\ell-1} \dots v_2$ are two distinct paths from v_1 to v_2, which is a contradiction. $\square$

We are now in a position to prove the main result of this lecture.

Theorem 2 (Cayley's Theorem)**.** *For each $n \in \mathbb{N}$, the number of trees on $[n]$ is n^{n-2}.*

Proof. Fix a positive integer n, and let t_n denote the number of trees on $[n]$. Consider the set $\mathscr{T}_n$ consisting of trees on $[n]$ such that each T in $\mathscr{T}_n$ has a distinguished pair of vertices $(b, e) \in V \times V$ (it may be that $b = e$). We proceed to show that $\mathscr{T}_n$ is in bijection with the set $\mathscr{F}_n$ consisting of all functions $f \colon [n] \to [n]$. Let $f \colon [n] \to [n]$ be a function, and let C be the set of elements in $[n]$ that are part of a cycle under the action of f, that is,

$$C := \{c \in [n] \mid f^m(c) = c \text{ for some } m \in \mathbb{N}\}.$$

Write $C = \{c_1, \dots, c_k\}$, where $c_1 < \cdots < c_k$. Observe that $C = \{f(c) \mid c \in C\}$. Now we produce an element of $T_f \in \mathscr{T}_n$ as follows. Let $[n]$ be the set of vertices of T_f. Then create a path with the elements in C by adding, for each $j \in [k-1]$, an edge between $f(c_i)$ and $f(c_{i+1})$. Then, for every element $v \in [n] \setminus C$, add an edge from v to $f(v)$. Finally, let $(f(c_1), f(c_k))$ be the distinguished pair of vertices of T_f.

Claim 1. $T_f \in \mathscr{T}_n$. Observe that for each $v \notin C$, there is exists $m \in \mathbb{N}$ such that $f^m(v) \in C$ as, otherwise, there would be a cycle (under the action of f) disjoint from C. Thus, any $v \in V(T_f) \setminus C$ is connected to a vertex of C. This, along with the fact that the vertices in C form a path in T_f, ensures that T_f is connected. To verify that T_f has no cycles, first note that any potential cycle in T_f must involve a vertex in C because otherwise we would have a an f-cycle not contained in C. Then if we had a cycle not contained in C, there would be a path $w_1 v_1 v_2 \dots v_\ell w_\ell$ in T_f, where $w_1, w_\ell \in C$ and $v_1, \dots, v_\ell \in [n] \setminus C$, and so $f(v_\ell) = w_\ell$, which implies that $f(v_{\ell-1}) = v_\ell$, and so we would obtain that $f(v_1) = v_2$, which generates a conflict with the fact that $f(v_1) = w_1$. Hence every potential cycle of T_f must involve only vertices in C, and the fact that C is a path allows us to conclude that T_f has no cycles. Thus, T_f is a tree.

every edge of T_f has the form $(v, f(v))$ for some $v \in V$. Thus, the edges of any potential cycle of T_f would be edges connecting the vertices in C, but all the edges connecting any two vertices of C in T_f form a path, which is free of cycles. Hence T_f is a tree with the distinguished pair $(f(c_1), f(c_k))$, which means that $T_f \in \mathscr{T}_n$.

Claim 2. The map $f \mapsto T_f$ is a bijection. Suppose that T is a tree on $[n]$ with distinguished pair (b, e). We will construct a map $f_T \colon [n] \to [n]$ as follows. Since T is a tree, there is a unique path $P := f_1 f_2 \dots f_k$ from b to e. Let $c_1, \dots, c_k$ be a rearrangement of $f_1, \dots, f_k$ such that $c_1 < \cdots < c_k$, and define $f_T(c_i) = f_i$. If a vertex $w \in [n]$ is not part of the path P, set $f_T(w) = v$, where v is the only adjacent vertex to w in the unique path from w to P (note that there is only one path from w to P because T does not contain any cycles). It is easy to check that $T \mapsto f_T$ is the inverse function of $f \mapsto T_f$.

Therefore $|\mathscr{T}_n| = n^n$, which is the number of functions from $[n]$ to $[n]$. On the other hand, $|\mathscr{T}_n| = n^2 t_n$ as, by definition, elements of $\mathscr{T}_n$ are pairs $(T, (b, e))$, where T is a tree on $[n]$ and $(b, e) \in V(T)^2$. Thus, we conclude that $t_n = n^{n-2}$. □

A *rooted tree* on a nonempty set V is a pair (T, v), where T is a tree with $V(T) = V$ and $v \in V$. A *rooted forest* on V is a forest with set of vertices V whose connected components are rooted trees.

Corollary 3. *For every $n \in \mathbb{N}$, there are $(n+1)^{n-1}$ rooted forests on $[n]$, that is, forests on $[n]$.*

Practice Exercises

Exercise 1. *Prove Corollary 3.*

Exercise 2. *A function* $f\colon [n] \to [n]$ *is called* acyclic *provided that the action of* f *on* $[n]$ *does not generate any cycle of length larger than* 1. *Prove that there are exactly* $(n+1)^{n-1}$ *acyclic functions from* $[n]$ *to* $[n]$.

References

[1] M. Bóna: *A Walk Through Combinatorics: An Introduction to Enumeration and Graph Theory* (Fourth Edition), World Scientific, New Jersey, 2017.

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