MIT 18.211: COMBINATORIAL ANALYSIS

FELIX GOTTI

Lecture 23: Intro to Trees

In this lecture, we introduce trees and discuss some basic related properties.

Definition 1. A simple connected graph is called a *tree* if it does not contain any cycle.

A connected simple graph G is said to be *minimally connected* if any graph obtained from G by deleting an edge is disconnected. As the following proposition indicates, trees are precisely those graphs that are minimally connected.

Proposition 2. *A graph is a tree if and only if it is minimally connected.*

Proof. Let G be a graph.

To prove the direct implication, assume that G is a tree and suppose, towards a contradiction, that G is not minimally connected. Then G contains an edge $e = vw$ that we can delete to obtain a connected graph G'. Since G' is connected, there must be a path $v_1v_2 \dots v_\ell$ in G' connecting v to w (in particular, $v_1 = v$ and $v_\ell = w$). Since $vw \notin V(G')$, we see that $v_1v_2 \dots v_\ell v_1$ is a cycle in G, which yields the desired contradiction.

To establish the converse, assume that G is minimally connected and suppose, by way of contradiction, that G is not a tree. Then G contains a cycle $v_1v_2 \dots v_\ell v_1$. Let G' be the graph that we obtain from G by removing the edge v_1v_2. We claim that G' is a connected graph. To prove this, let v and w be two distinct vertices of G', and let P be a path in G from v to w, which must exist because G is connected. If v_1v_2 is not an edge of P, then P is a path from v to w in G'. Otherwise, we can replace the edge v_1v_2 in P by the path $v_1v_\ell v_{\ell-1} \dots v_2$ to turn P into a walk from v to w in G'. Among all such walks, pick one $P_0$of minimum length. It is clear that P_0 is a path from v to w in G'. Hence we conclude that G' is connected, which contradicts that G is minimally connected. □

As we show in the next proposition a tree with n vertices must contain precisely $n-1$ edges.

Proposition 3. *If G is a tree, then $|E(G)| = |V(G)| - 1$.*

Proof. We will argue that for every tree G, the equality $|E(G)| = |V(G)| - 1$ by using induction on $|V(G)|$. If $|V(G)| = 1$, then G is the graph consisting of one vertex and no edges, and so the equality $|E(G)| = |V(G)| - 1$ holds immediately. Assume that every tree with $n \in \mathbb{N}$ vertices has precisely $n-1$ edges, and let G be a tree with $|V(G)| = n+1$. Let P be a path in G with maximum number of edges. Since G is connected and $|V(G)| \geq 2$, it follows that $\deg v \geq 1$ for every $v \in V(G)$. Thus, P has at least two vertices, and the maximality of P implies that the first and last vertices of P have degree 1. Take $v \in V(G)$ such that $\deg v = 1$. Observe that the graph G' we obtain from G after removing v (and the only edge in G that is adjacent to v) is a tree with $|V(G')| = n$. By our induction hypothesis, $|E(G')| = n-1$ and, therefore $|E(G)| = |E(G')| + 1 = n = |V(G)| - 1$. □

If we drop connectedness from the definition of a tree, we obtain a class of graphs that are also quite relevant. They are called forests.

Definition 4. A *forest* is a graph whose connected components are trees.

We can generalize Proposition 3 to forests as follows.

Proposition 5. *Let G be a forest with exactly k connected components. Then $|E(G)| = |V(G)| - k$.*

Proof. [1] Let $T_1, \dots, T_k$ be the connected components of G. Now construct a graph G' from G be adding $k-1$ edges $e_1, \dots, e_{k-1}$ such that the edge e_i connect a vertex of T_i with a vertex of T_{i+1}. It is clear that G' is a connected graph with no cycles. Hence G' is a tree, and it follows from Proposition 3 that $|E(G')| = |V(G')| - 1$. This implies that $|E(G)| = |E(G')| - (k-1) = |V(G')| - k = |V(G)| - k$. □

Practice Exercises

Exercise 1. [1, Exercise 10.29] *Prove that in any tree, any two longest paths cross each other.*

Exercise 2. [1, Exercise 10.31] *Let T be a tree, and let $P(T)$ be the induced subgraph of T whose vertices of T in the intersection of all longest paths of T. Describe the graph $P(T)$.*

Exercise 3. *Let G be a connected graph satisfying $|E(G)| = |V(G)| - 1$. Prove that G is a tree.*

[1] Observe that this prove also follows mimicking the inductive proof given for trees.

References

[1] M. Bóna: *A Walk Through Combinatorics: An Introduction to Enumeration and Graph Theory* (Fourth Edition), World Scientific, New Jersey, 2017.

Department of Mathematics, MIT, Cambridge, MA 02139
Email address: fgotti@mit.edu