MIT 18.211: COMBINATORIAL ANALYSIS

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Lecture 22: Hamiltonian Cycles and Paths

In this lecture, we discuss the notions of Hamiltonian cycles and paths in the context of both undirected and directed graphs.

Hamiltonian Cycles and Paths. Let G be a graph. A *cycle* in G is a closed trail that only repeats the first and last vertices. A *Hamiltonian cycle* (resp., a *Hamiltonian path*) in G is a cycle (resp., a path) that visits all the vertices of G. As for (closed) Eulerian trails, we are interested in the question of whether a given graph has a Hamiltonian cycle/path.

Definition 1. A simple graph that has a Hamiltonian cycle is called a *Hamiltonian graph*.

We observe that not every graph is Hamiltonian; for instance, it is clear that a disconnected graph cannot contain any Hamiltonian cycle/path. There are also connected graphs that are not Hamiltonian. For example, if a connected graph has a a vertex of degree one, then it cannot be Hamiltonian.

Example 2. A cycle on n vertices has exactly one cycle, which is a Hamiltonian cycle. Then cycles are Hamiltonian graphs.

Example 3. The complete graph K_n is Hamiltonian if and only if $n \geq 3$.

The following proposition provides a condition under which we can always guarantee that a graph is Hamiltonian.

Proposition 4. *Fix $n \in \mathbb{N}$ with $n \geq 3$, and let $G = (V, E)$ be a simple graph with $|V| \geq n$. If $\deg v \geq n/2$ for all $v \in V$, then G is Hamiltonian.*

Proof. Suppose, by way of contradiction, that there exists a graph satisfying the hypothesis of the proposition that is not Hamiltonian. Among all such graph assume that G is one which minimizes the number of vertices. After adding as many edges as necessary, we can replace G by a graph $G' = (V, E')$ with no Hamiltonian cycles and $\deg v \geq n/2$ for all $v \in V$, which also satisfies the condition that any graph obtained from G' by adding a new edge will contain a Hamiltonian cycle.

Since $n \geq 3$ and G' does not have any Hamiltonian cycle, we can pick two distinct vertices $v, w \in V$ that are not adjacent. As including an edge between v and w would create a Hamiltonian cycle, there must be a Hamiltonian path from v to w, namely, $v_1v_2\dots v_n$ with $v_1 = v$ and $v_n = w$. Now consider the sets

$$X := \{i \in [\![2, n-1]\!] \mid v_iw \in E'\} \quad \text{and} \quad Y := \{i \in [\![2, n-1]\!] \mid v_{i+1}v \in E'\}.$$

Since $v = v_1, v_2, \dots, v_{n-1}, v_n = w$ is a Hamiltonian path and the vertices v and w are not adjacent, $|X| = \deg w$ and $|Y| = \deg v$. Thus,

$$|X \cap Y| = |X| + |Y| - |X \cup Y| = \deg w + \deg v - |X \cup Y| \ge n - |X \cup Y| \ge 2,$$

where the last inequality follows from the fact that $X \cup Y \subseteq [\![2, n-1]\!]$ and, therefore, $|X \cup Y| \le n-2$. Thus, $X \cap Y$ is nonempty. Now one can take $j \in X \cap Y$ to obtain the following Hamiltonian cycle: $v_1v_2\dots v_jv_nv_{n-1}v_{j+1}v_1$, which is a contradiction. □

Directed Graphs. Now we consider Hamiltonian cycles in directed graphs. A directed graph is called a *tournament* if there is a directed edge between any two vertices. Observe that a directed graph (V, E) is a tournament if and only if it contains $\binom{n}{2}$ edges, where $n = |V|$.

Proposition 5. *Every tournament has a Hamiltonian path.*

Proof. We proceed by induction on the number of vertices of a tournament. It is clear that a tournament with two (or one) vertices has a Hamiltonian path. Suppose that every tournament with $n \ge 2$ vertices has a Hamiltonian path. Let $G = (V, E)$ be a tournament with $n+1$ vertices. Fix $v_{n+1} \in V$, and let H be the subgraph of G induced by the vertices $V \setminus \{v_{n+1}\}$. It is clear that H is a tournament, and so it follows from our induction hypothesis that H has a Hamiltonian path, namely, $C := v_1v_2, v_2v_3, \dots, v_{n-1}v_n$. If $\operatorname{outdeg} v_{n+1} = 0$, then $v_nv_{n+1} \in E$, and so $v_1v_2, v_2v_3, \dots, v_nv_{n+1}$ is a Hamiltonian path of G. Suppose, therefore, that $\operatorname{outdeg}, v_{n+1} \ge 1$, and let $j \in [\![1, n]\!]$ be the smallest index such that $v_{n+1}v_j \in E$. If $j = 1$, then $v_{n+1}v_1, v_1v_2, \dots, v_{n-1}v_n$ is a Hamiltonian path. Otherwise, the minimality of j guarantees that $v_{j-1}v_{n+1} \in E$, and so $v_1v_2, v_2v_3, \dots, v_{j-1}v_{n+1}, v_{n+1}v_j, v_jv_{j+1}, \dots, v_{n-1}v_n$ is a Hamiltonian path. □

Practice Exercises

Exercise 1. *Let G be a simple graph with 10 vertices and 28 edges. Prove that G contains a cycle of length 4.*

Exercise 2. [1, Exercise 9.40] *How many Hamiltonian cycles does K_n have?*

References

[1] M. Bóna: *A Walk Through Combinatorics: An Introduction to Enumeration and Graph Theory* (Fourth Edition), World Scientific, New Jersey, 2017.

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