Lecture 21: Introduction to Graphs and Eulerian Trails

In this lecture, we begin our journey through graph theory.

Definition 1. A simple graph is a pair \((V, E)\), where \(V\) is a finite nonempty set and \(E\) is a set consisting of 2-subsets of \(V\). The elements of \(V\) are called vertices while the elements of \(E\) are called edges.

In coming lectures, we will also consider general graphs \((V, E)\), where we allow multiple edges between two given vertices (that is, \(E\) is a multiset instead of a set) and we also allow loops, which are edges from a vertex to itself (that is, \(E\) may also contain 1-subsets of \(V\)). We will formally define general graphs and other types of graphs when we need them.

Let \(G = (V, E)\) be a simple graph. It is convenient to set \(V(G) := V\) and \(E(G) := E\). We often denote an edge \(\{v, w\}\) of \(G\) simply by \(vw\). It is clear that \(vw\) and \(wv\) both denote the same edge. If \(vw\) is an edge of \(G\), then we say that the vertices \(v\) and \(w\) are adjacent vertices. For any vertex \(v \in V\), the degree of \(v\), denoted by \(\deg v\), is the number of edges connected to \(v\), that is,

\[
\deg v = |\{e \in E \mid v \in e\}|.
\]

Example 2. For every \(n \in \mathbb{N}\) with \(n \geq 3\), there is a unique graph \(C_n\) satisfying that \(\deg v = 2\) for every \(v \in V\). We will formally define what we mean by “unique” in coming lectures. The graph \(C_n\) looks like a polygon of \(n\) vertices whose edges are the sides of the polygon. It is called the cycle on \(n\) vertices.

Example 3. For every \(n \in \mathbb{N}\), there is a unique graph having \(n\) vertices and no edges. In this graph no two vertices are adjacent; it is sometimes called the trivial graph of \(n\) vertices. On the other hand, there is a unique graph having \(n\) vertices, where any two distinct vertices are adjacent. This is called the complete graph on \(n\) vertices, and it is denoted by \(K_n\). Observe that \(K_n\) has precisely \(\binom{n}{2}\) edges.

The following proposition provides a restriction on the degrees of the vertices of a graph.

Proposition 4. Every graph contains an even number of vertices of odd degree.
Proof. Let \( G = (V, E) \) be a graph. It suffices to observe that every edge of \( G \) (if some) contributes 1 towards the degree of each of the two vertices it is connected to. Therefore

\[
\sum_{v \in V} \deg v = 2|E|.
\]

Since the sum of all the degrees of vertices of \( G \) is even, there must be an even number of vertices of \( G \) with odd degrees. \( \square \)

A sequence of edges of the form \( v_1v_2, v_2v_3, \ldots, v_{\ell}v_{\ell+1} \), where \( v_1, \ldots, v_{\ell+1} \in V(G) \) is called a walk of length \( \ell \) and, if \( v_iv_{i+1} \neq v_jv_{j+1} \) for any distinct \( i, j \in [\ell] \), then we call \( v_1v_2, v_2v_3, \ldots, v_{\ell}v_{\ell+1} \) a trail of length \( \ell \). We often denote a walk/trail \( v_1v_2, \ldots, v_{\ell}v_{\ell+1} \) simply by \( v_1v_2 \ldots v_{\ell+1} \). Clearly, the walks/trails of length 1 are precisely the edges. If a trail \( v_1v_2v_3 \ldots v_{\ell+1} \) satisfies that \( v_{\ell+1} = v_1 \), then we call it a closed trail or a circuit (in this case, note that \( \ell \geq 3 \)). A trail (resp., circuit) that uses all the edges of the graph is called an Eulerian trail (resp., Eulerian circuit).

If a trail \( v_1v_2 \ldots v_{\ell+1} \) satisfies that \( v_i \neq v_j \) for any \( i \neq j \), then it is called a path. A subgraph of \( G \) is a graph \( (V', E') \) such that \( V' \subseteq V \) and \( E' \subseteq E \). A subgraph \( (V', E') \) of \( G \) is called an induced subgraph provided that for any \( v, w \in V' \) with \( v \neq w \), if \( vw \in E \), then \( vw \in E' \). We say that the graph \( G \) is connected if any two distinct vertices of \( G \) can be connected by a path, that is, for any \( v, w \in V \) with \( v \neq w \) there exists a path \( v_1v_2 \ldots v_{\ell+1} \) such that \( v_1 = v \) and \( v_{\ell+1} = w \). A graph that is not connected is called disconnected. A connected component \( C \) of \( G \) is a maximal connected subgraph of \( G \), that is, \( C \) is a connected subgraph of \( G \) and if \( C \) is a subgraph of a connected subgraph \( C' \) of \( G \), then \( C' = C \).

**Theorem 5.** A connected graph \( G \) has an Eulerian circuit if and only if every vertex of \( G \) has even degree.

Proof. Let \( G = (V, E) \) be a connected graph. If \( G \) has only one vertex, the statement of the theorem follows trivially. So we assume that \( |V| \geq 2 \). Since \( G \) is connected, \( |E| \geq 1 \).

For the direct implication, suppose that \( C := v_1v_2, v_2v_3, \ldots, v_{\ell}v_1 \) is an Eulerian circuit of \( G \). If we travel through \( C \) departing from \( v_1 \), then we will pass through every edge of \( G \) exactly once. Moreover, if \( m_i \) is the number of times we pass by \( v_i \) for \( i \neq 1 \), then \( \deg v_i = 2m_i \) as in order to touch any of these vertices we have to travel through two incident edges we haven’t encountered before. A similar argument shows that \( \deg v_1 = 2m_1 \) if \( m_1 \) is the number of times we visit \( v_1 \) after departure, as the last edge of \( C \) that is used to arrive to \( v_1 \) for last time can be matched with the first edge, the one we used to depart from \( v_1 \).

For the converse, suppose that every vertex of \( G \) has even degree. We proceed by induction on the number \( n \) of edges. The implication trivially holds when \( n = 0 \) (or \( n = 3 \)). Suppose that \( |E| = n > 3 \) and also that the statement of the theorem
holds for any connected graph with less than \( n \) edges. Choose a vertex \( v_1 \in V \) and start traveling from \( v_1 \) through consecutive edges of \( G \) without repeating any of them until encountering a vertex from which we cannot continue traveling because all the edges connected to this vertex have been already used. This will give a trail \( T := v_1v_2, v_2v_3, \ldots, v_tv_{t+1} \). If \( v_{t+1} \in V \setminus \{v_1\} \), then \( v_{t+1} \) would have been connected to an odd number of edges in \( T \) and, given \( \deg v_{t+1} \) is even, we would have been able to continue traveling. Thus, \( v_{t+1} = v_1 \). We have proved that \( G \) contains a closed trail. Among all such closed trails, let \( C \) be one with maximum number of edges.

We claim that \( C \) is an Eulerian circuit. Suppose, by way of contradiction, that this is not the case. Consider the subgraph \( G' := (V, E') \) of \( G \), where \( E' = E \setminus C \). Since \( C \) is not an Eulerian circuit, \( E' \) is nonempty. Take \( e \in E' \), and let \( H \) be the connected component of \( G' \) containing \( e \). Since \( H \) is connected and \( |E'| < |E| \), it follows from the induction hypothesis that \( H \) has an Eulerian circuit \( C' \). Observe that one of the edges of \( C \) must be connected to a vertex \( x \) in \( H \), as otherwise \( G \) would be disconnected. Now notice that the closed trail that results from concatenating \( C \) and \( C' \) via \( x \) has more edges than \( C \). This contradicts, however, the maximality of \( C \). As a consequence, \( C \) must be an Eulerian circuit. \( \square \)

**Corollary 6.** A connected graph has an Eulerian trail if and only if it contains at most two vertices with odd degrees.

**Proof.** Exercise. \( \square \)

**Directed Graphs.** If we think of the edges of a graph as arrows with certain orientations, then we obtain a more convenient structure to model certain situations. For instance, we can model a (round-robin) tournaments by considering competitors to be the vertices and matches to be the arrows, each of them directed towards the winner. This gives a natural variation of the definition of a graph introduced in the previous subsection.

**Definition 7.** A (simple) directed graph is a pair \( (V, E) \), where \( V \) is a finite set and \( E \) is a subset of \( V \times V \) satisfying the following two conditions:

1. \( (v, v) \notin E \) for any \( v \in V \) and
2. \( |\{(v, w), (w, v)\} \cap E| \leq 1 \) for all \( v, w \in V \).

We illustrate a directed graphs in the same way as we do with (undirected) graphs, but we represent an edge \( vw \) by drawing an arrow from \( v \) to \( w \). Condition (1) in the above definition means that there is no arrow directed from a vertex to itself while condition (2) means that two arrows never create a directed cycle.

Let \( G = (V, E) \) be a directed graph. For any \( v \in V \), the in-degree of \( v \), denoted by \( \deg v \), is the number of edges incident and directed to \( v \). In a similar manner, we define the out-degree of a vertex \( v \) and we denote it by \( \deg v \). We say that \( G \) is balanced provided that \( \deg v = \deg v \) for all \( v \in V \).
We can define walks, (Eulerian) trails, (Eulerian) circuits, and paths for directed graphs in the same way we did it for (undirected) graphs. We say that a directed graph $G$ is strongly connected if for any two distinct vertices $v$ and $w$ of $G$, we can find a (directed) path from $v$ to $w$.

For directed graphs, we are also interested in the existence of Eulerian circuits/trails. For Eulerian circuits, the following result is parallel to that we have proved for undirected graphs.

**Theorem 8.** A directed graph has an Eulerian circuit if and only if it is a balanced strongly connected graph.

**Proof.** The direct implication is obvious as when we travel through an Eulerian circuit every time we enter a vertex we have to leave, going through two non-used edges. The reverse implication follows by mimicking the proof of the similar statement for undirected graphs (see the proof of Theorem 5). □

**Practice Exercises**

**Exercise 1.** [1, Exercise 9.17] Is there a simple graph on 6 vertices with ordered degree sequence $4, 4, 4, 2, 1, 1$?

**Exercise 2.** For a simple graph $G = (V, E)$, we define $G^*$ to be the graph with set of vertices $V$ and an edge between $v$ and $w$ if and only if $vw \notin E$. For any simple graph $G$, prove that either $G$ or $G^*$ is connected.

**Exercise 3.** Prove Corollary 6.

**Exercise 4.** For which $n \in \mathbb{N}$, does the complete graph $K_n$ have an Eulerian trail/circuit?

**Exercise 5.** Can we always add edges to a simple graph to obtain another simple graph with an Eulerian circuit?

**References**


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