MIT 18.211: COMBINATORIAL ANALYSIS

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Lecture 21: Introduction to Graphs and Eulerian Trails

In this lecture, we begin our journey through graph theory.

Definition 1. A *simple graph* is a pair (V, E), where V is a finite nonempty set and E is a set consisting of 2-subsets of V. The elements of V are called *vertices* while the elements of E are called *edges*.

In coming lectures, we will also consider general graphs (V, E), where we allow multiple edges between two given vertices (that is, E is a multiset instead of a set) and we also allow loops, which are edges from a vertex to itself (that is, E may also contain 1-subsets of V). We will formally define general graphs and other types of graphs when we need them.

Let $G = (V, E)$ be a simple graph. It is convenient to set $V(G) := V$ and $E(G) := E$. We often denote an edge $\{v, w\}$ of G simply by vw. It is clear that vw and wv both denote the same edge. If vw is an edge of G, then we say that the vertices v and w are *adjacent* vertices. For any vertex $v \in V$, the *degree* of v, denoted by $\deg v$, is the number of edges connected to v, that is,

$$\deg v = |\{e \in E \mid v \in e\}|.$$

Example 2. For every $n \in \mathbb{N}$ with $n \geq 3$, there is a unique graph C_n satisfying that $\deg v = 2$ for every $v \in V$. We will formally define what we mean by "unique" in coming lectures. The graph C_n looks like a polygon of n vertices whose edges are the sides of the polygon. It is called the *cycle* on n vertices.

Example 3. For every $n \in \mathbb{N}$, there is a unique graph having n vertices and no edges. In this graph no two vertices are adjacent; it is sometimes called the trivial graph of n vertices. On the other hand, there is a unique graph having n vertices, where any two distinct vertices are adjacent. This is called the *complete* graph on n vertices, and it is denoted by K_n. Observe that K_n has precisely $\binom{n}{2}$ edges.

The following proposition provides a restriction on the degrees of the vertices of a graph.

Proposition 4. *Every graph contains an even number of vertices of odd degree.*

Proof. Let $G = (V, E)$ be a graph. It suffices to observe that every edge of G (if some) contributes 1 towards the degree of each of the two vertices it is connected to. Therefore

$$\sum_{v \in V} \deg v = 2|E|.$$

Since the sum of all the degrees of vertices of G is even, there must be an even number of vertices of G with odd degrees. $\square$

A sequence of edges of the form $v_1v_2, v_2v_3, \dots, v_\ell v_{\ell+1}$, where $v_1, \dots, v_{\ell+1} \in V(G)$ is called a *walk* of *length* ℓ and, if $v_iv_{i+1} \neq v_jv_{j+1}$ for any distinct $i, j \in [\ell]$, then we call $v_1v_2, v_2v_3, \dots, v_\ell v_{\ell+1}$ a *trail* of *length* ℓ. We often denote a walk/trail $v_1v_2, \dots, v_\ell v_{\ell+1}$ simply by $v_1v_2 \dots v_{\ell+1}$. Clearly, the walks/trails of length 1 are precisely the edges. If a trail $v_1v_2v_2 \dots v_{\ell+1}$ satisfies that $v_{\ell+1} = v_1$, then we call it a *closed trail* or a *circuit* (in this case, note that $\ell \ge 3$). A trail (resp., circuit) that uses all the edges of the graph is called an *Eulerian trail* (resp., *Eulerian circuit*).

If a trail $v_1v_2 \dots v_{\ell+1}$ satisfies that $v_i \neq v_j$ for any $i \neq j$, then it is called a *path*. A *subgraph* of G is a graph (V', E') such that $V' \subseteq V$ and $E' \subseteq E$. A subgraph (V', E') of G is called an *induced subgraph* provided that for any $v, w \in V'$ with $v \neq w$, if $vw \in E$, then $vw \in E'$. We say that the graph G is *connected* if any two distinct vertices of G can be connected by a path, that is, for any $v, w \in V$ with $v \neq w$ there exists a path $v_1v_2 \dots v_{\ell+1}$ such that $v_1 = v$ and $v_{\ell+1} = w$. A graph that is not connected is called *disconnected*. A *connected component* C of G is a maximal connected subgraph of G, that is, C is a connected subgraph of G and if C is a subgraph of a connected subgraph C' of G, then $C' = C$.

Theorem 5. *A connected graph G has an Eulerian circuit if and only if every vertex of G has even degree.*

Proof. Let $G = (V, E)$ be a connected graph. If G has only one vertex, the statement of the theorem follows trivially. So we assume that $|V| \ge 2$. Since G is connected, $|E| \ge 1$.

For the direct implication, suppose that $C := v_1v_2, v_2v_3, \dots, v_\ell v_1$ is an Eulearian circuit of G. If we travel through C departing from v_1, then we will pass through every edge of G exactly once. Moreover, if m_i is the number of times we pass by v_i for $i \neq 1$, then $\deg v_i = 2m_i$ as in order to touch any of these vertices we have to travel through two incident edges we haven't encountered before. A similar argument shows that $\deg v_1 = 2m_1$ if m_1 is the number of times we visit v_1 after departure, as the last edge of C that is used to arrive to v_1 for last time can be matched with the first edge, the one we used to depart from v_1.

For the converse, suppose that every vertex of G has even degree. We proceed by induction on the number n of edges. The implication trivially holds when $n = 0$ (or $n = 3$). Suppose that $|E| = n > 3$ and also that the statement of the theorem holds for any connected graph with less than n edges. Choose a vertex $v_1 \in V$ and start traveling from v_1 through consecutive edges of G without repeating any of them until encountering a vertex from which we cannot continue traveling because all the edges connected to this vertex have been already used. This will give a trail $T := v_1v_2, v_2v_3, \dots, v_\ell v_{\ell+1}$. If $v_{\ell+1} \in V \setminus \{v_1\}$, then $v_{\ell+1}$ would have been connected to an odd number of edges in T and, given $\deg v_{\ell+1}$ is even, we would have been able to continue traveling. Thus, $v_{\ell+1} = v_1$. We have proved that G contains a closed trail. Among all such closed trails, let C be one with maximum number of edges.

We claim that C is an Eulerian circuit. Suppose, by way of contradiction, that this is not the case. Consider the subgraph $G' := (V, E')$ of G, where $E' = E \setminus C$. Since C is not an Eulearian circuit, E' is nonempty. Take $e \in E'$, and let H be the connected component of G' containing e. Since H is connected and $|E'| < |E|$, it follows from the induction hypothesis that H has an Eulearian circuit C'. Observe that one of the edges of C must be connected to a vertex x in H, as otherwise G would be disconnected. Now notice that the closed trail that results from concatenating C and C' via x has more edges than C. This contradicts, however, the maximality of C. As a consequence, C must be an Eulerian circuit. $\square$

Corollary 6. *A connected graph has an Eulerian trail if and only if it contains at most two vertices with odd degrees.*

Proof. Exercise. $\square$

Directed Graphs. If we think of the edges of a graph as arrows with certain orientations, then we obtain a more convenient structure to model certain situations. For instance, we can model a (round-robin) tournaments by considering competitors to be the vertices and matches to be the arrows, each of them directed towards the winner. This gives a natural variation of the definition of a graph introduced in the previous subsection.

Definition 7. A (*simple*) *directed graph* is a pair (V, E), where V is a finite set and E is a subset of $V \times V$ satisfying the following two conditions:

(1) $(v, v) \notin E$ for any $v \in V$ and
(2) $|\{(v, w), (w, v)\} \cap E| \leq 1$ for all $v, w \in V$.

We illustrate a directed graphs in the same way as we do with (undirected) graphs, but we represent an edge vw by drawing an arrow from v to w. Condition (1) in the above definition means that there is no arrow directed from a vertex to itself while condition (2) means that two arrows never create a directed cycle.

Let $G = (V, E)$ be a directed graph. For any $v \in V$, the *in-degree* of v, denoted by $\operatorname{indeg} v$, is the number of edges incident and directed to v. In a similar manner, we define the *out-degree* of a vertex v and we denote it by $\operatorname{outdeg} v$. We say that G is *balanced* provided that $\operatorname{indeg} v = \operatorname{outdeg} v$ for all $v \in V$.

We can define walks, (Eulerian) trails, (Eulerian) circuits, and paths for directed graphs in the same way we did it for (undirected) graphs. We say that a directed graph G is *strongly connected* if for any two distinct vertices v and w of G, we can find a (directed) path from v to w.

For directed graphs, we are also interested in the existence of Eulerian circuits/trails. For Eulerian circuits, the following result is parallel to that we have proved for undirected graphs.

Theorem 8. *A directed graph has an Eulerian circuit if and only if it is a balanced strongly connected graph.*

Proof. The direct implication is obvious as when we travel through an Eulerian circuit every time we enter a vertex we have to leave, going through two non-used edges. The reverse implication follows by mimicking the proof of the similar statement for undirected graphs (see the proof of Theorem 5). □

Practice Exercises

Exercise 1. [1, Exercise 9.17] *Is there a simple graph on* 6 *vertices with ordered degree sequence* $4, 4, 4, 2, 1, 1$*?*

Exercise 2. *For a simple graph $G = (V, E)$, we define G^* to be the graph with set of vertices V and an edge between v and w if and only if $vw \notin E$. For any simple graph G, prove that either G or G^* is connected.*

Exercise 3. *Prove Corollary 6.*

Exercise 4. *For which $n \in \mathbb{N}$, does the complete graph K_n have an Eulerian trail/circuit?*

Exercise 5. *Can we always add edges to a simple graph to obtain another simple graph with an Eulerian circuit?*

References

[1] M. Bóna: *A Walk Through Combinatorics: An Introduction to Enumeration and Graph Theory* (Fourth Edition), World Scientific, New Jersey, 2017.

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