MIT 18.211: COMBINATORIAL ANALYSIS

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Lecture 2: Mathematical Induction

Mathematical induction is a technique used to prove that a certain property holds for every positive integer (from one point on).

Principle of Mathematical Induction. For each (positive) integer n, let $P(n)$ be a statement that depends on n such that the following conditions hold:

(1) $P(n_0)$ is true for some (positive) integer n_0 and

(2) $P(n)$ implies $P(n+1)$ for every integer $n \geq n_0$.

Then $P(n)$ is true for every integer $n \geq n_0$.

With notation as before, step (1) is called the *base case* and step (2) is called the *induction step*. In the induction step, $P(n)$ is often called the *induction hypothesis*. Let us take a look at some scenarios where the principle of mathematical induction is an effective tool.

Example 1. Let us argue, using mathematical induction, the following formula for the sum of the squares of the first n positive integers:

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}. \tag{0.1}$$

Let $P(n)$ be the equality in (0.1). For the base case, it suffices to observe that when $n = 1$, both sides of (0.1) equal 1, and so $P(1)$ is true. For the induction step, assume that $P(n)$ is true for certain $n \in \mathbb{N}$. Then

$$\begin{aligned} 1^2 + 2^2 + \cdots + (n+1)^2 &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\ &= (n+1)\Big(\frac{2n^2+n}{6} + \frac{6n+6}{6}\Big) \\ &= \frac{(n+1)(n+2)(2n+3)}{6}, \end{aligned}$$

where the first equality holds because $P(n)$ is true. Thus, we have inferred $P(n+1)$ from $P(n)$ and so, by virtue of the principle of mathematical induction, we obtain that $P(n)$ is true for every $n \in \mathbb{N}$.

Example 2. It turns out that 7 divides $5^{2n+1} + 2^{2n+1}$ for every $n \in \mathbb{N}_0$. Well, let us show this by using induction. When $n = 0$, we see that $5^{2n+1} + 2^{2n+1} = 7$, and so it is divisible by 7. Suppose now that 7 divides $5^{2n+1} + 2^{2n+1}$ for some nonnegative integer n. Then 7 divides $5^2 \cdot 5^{2n+1} + (2^2 + 21) \cdot 2^{2n+1} = 5^{2(n+1)+1} + 2^{2(n+1)+1} + 21 \cdot 2^{2n+1}$. As $21 \cdot 2^{2n+1}$ is divisible by 7, we obtain that 7 divides $5^{2(n+1)+1} + 2^{2(n+1)+1}$. Hence it follows by mathematical induction that 7 divides $5^{2n+1} + 2^{2n+1}$ for every $n \in \mathbb{N}_0$.

Example 3. For a positive integer n, consider $3n$ points in the plane such that no three of them are collinear. Let us argue that we can form n disjoint triangles whose vertices are the $3n$ given points. When $n = 1$, we can form only one triangle, which trivially satisfies the desired condition. Suppose now that the statement to be proved holds for certain $n \in \mathbb{N}$, and let us argue that it must also hold for $n + 1$. To do so, let $P_1, \dots, P_{3(n+1)}$ denote the given points, and let P_i and P_j be two adjacent vertices of the convex hull of these $3n + 3$ points (i.e., the smallest polygon containing such points). Since no three of the points $P_1, \dots, P_{3n+3}$ are collinear, there is a unique $k \in [3n + 3] \setminus \{i, j\}$ minimizing the angle formed by P_i, P_j, and P_k. Let ℓ be the line determined by P_j and P_k, and observe that the minimality of the angle $\angle P_i P_j P_k$ ensures that all the points in the set $S := \{P_m \mid 1 \le m \le 3n + 3\} \setminus \{P_i, P_j, P_k\}$ lie in the half-plane determined by the line ℓ that does not contain P_i. The induction hypothesis allows us to form n disjoint triangles whose vertices are the points in S. Each of these n triangles, being contained in the half-space determined by ℓ opposite to P_i, must be disjoint from the triangle $\triangle P_i P_j P_k$. Hence we can take $\triangle P_i P_j P_k$ to be our $(n + 1)$-th triangle.

The following variation of the principle of mathematical induction, called strong induction, is usually convenient.

Strong Induction. For each (positive) integer n, let $P(n)$ be a statement that depends on n such that the following conditions hold:

(1) $P(n_0)$ is true for some (positive) integer n_0 and

(2) $P(n_0), \dots, P(n)$ implies $P(n + 1)$ for every integer $n \ge n_0$.

Then $P(n)$ is true for every integer $n \ge n_0$.

To illustrate an application of the strong mathematical induction principle, let us prove the (existential part of the) Fundamental Theorem of Arithmetic.

Example 4. We know that every $n \in \mathbb{N}$ with $n \ge 2$ can be factored into primes. Let's prove it! When $n = 2$, it is its own factorization into primes. Now fix $n \ge 2$ and suppose that every positive integer between 2 and n (including both) can be factored into primes. Let us argue that $n + 1$ can also be factored into primes. If $n + 1$ is prime, then it is its own factorization into primes. Otherwise $n + 1$ is composite, and so $n + 1 = ab$ for some positive integers a and b with $2 \le a, b \le n$. By our (strong) induction hypothesis, $a = p_1 \cdots p_k$ and $b = p_{k+1} \cdots p_m$ for some primes $p_1, \ldots, p_m$. Therefore $n+1$ factors into primes as $n+1 = p_1 \cdots p_m$. Hence we have proved that every integer greater than 1 factors into primes.

Practice Exercises

Exercise 1. *Prove that*

$$\sum_{k=1}^{n} k^3 = \Big(\sum_{k=1}^{n} k\Big)^2 = \Big(\frac{n(n+1)}{2}\Big)^2$$

for every $n \in \mathbb{N}$.

Exercise 2. [1, Exercise 1.2] *At a tennis tournament, every two players play against each other exactly once. After the tournament is over, each player lists the names of those he/she defeated and the names of those defeated by someone he/she defeated. Prove that there is one player who listed the names of everyone else.*

Exercise 3. *Let* r *be a nonzero real number such that* $r + r^{-1}$ *is an integer. Show that* $r^n + r^{-n}$ *is an integer for every* $n \in \mathbb{N}$.

Exercise 4. *Suppose that we divide the plane into regions using straight lines. Show that we can color each region either blue or red so that no two neighbor regions (i.e., two distinct regions having a common side) have the same color.*

References

[1] M. Bóna: *A Walk Through Combinatorics: An Introduction to Enumeration and Graph Theory* (Fourth Edition), World Scientific, New Jersey, 2017.

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