MIT 18.211: COMBINATORIAL ANALYSIS

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Lecture 18: Exponential Generating Functions I

There are many sequences that grow too quickly and, therefore, their generating functions cannot be expressed in closed form. For some of such sequences, it is convenient to use exponential generating functions instead of ordinary generating functions.

Definition 1. Let $(a_n)_{n\ge 0}$ be a sequence of real numbers. The *exponential generating function* of $(a_n)_{n\ge 0}$ is the formal series $\sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$.

Here are two very elementary but important examples.

Example 2. The generating function of the constant sequence whose terms are 1's is

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x.$$

Example 3. The exponential generating function for the sequence $(n!)_{n\ge 0}$ is

$$\sum_{n=0}^{\infty} n!\frac{x^n}{n!} = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

As for ordinary generating functions, we can differentiate, add, and multiply exponential generating functions.

Definition 4. If $F(x)$ and $G(x)$ are the exponential generating functions of $(a_n)_{n\ge 0}$ and $(b_n)_{n\ge 0}$, then we define

(1) $F(x) + G(x) = \sum_{n=0}^{\infty}(a_n + b_n)\frac{x^n}{n!}$,
(2) $F(x)G(x) = \sum_{n=0}^{\infty}\big(\sum_{k=0}^{n}\binom{n}{k}a_k b_{n-k}\big)\frac{x^n}{n!}$, and
(3) $F'(x) = \sum_{n=1}^{\infty} a_n \frac{x^{n-1}}{(n-1)!}$.

The following proposition yields a useful interpretation for the coefficients of the product of two exponential generating functions.

Proposition 5. *For $n \in \mathbb{N}_0$, let a_n and b_n be the numbers of ways to build certain α-structure and β-structure on an n-set, respectively. Let f_n be the number of ways to partition $[n]$ into two sets S and T and then place an α-structure on S and a β-structure on T. If $A(x)$, $B(x)$, and $F(x)$ are the exponential generating functions of $(a_n)_{n\ge 0}$, $(b_n)_{n\ge 0}$, and $(f_n)_{n\ge 0}$, respectively, then $F(x) = A(x)B(x)$.*

Proof. Observe that, for every $n \in \mathbb{N}_0$,

$$f_n = \sum_{S \subseteq [n]} a_{|S|} b_{n-|S|} = \sum_{k=0}^{n} \sum_{S \subseteq [n] : |S| = k} a_k b_{n-k} = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}.$$

Therefore

$$\begin{aligned} F(x) &= \sum_{n=0}^{\infty} f_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} \bigg(\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} \bigg) \frac{x^n}{n!} \\ &= \sum_{n=0}^{\infty} \bigg(\sum_{k=0}^{n} \frac{a_k}{k!} \frac{b_{n-k}}{(n-k)!} \bigg) x^n = A(x)B(x), \end{aligned}$$

where in the last equality we are using the formula for the product of two ordinary generating functions. □

Example 6. Let us find an explicit formula for the exponential generating function $B(x)$ of the sequence $(B(n))_{n \ge 0}$, where $B(n)$ denotes the n-th Bell's number. We have learned before that

$$B(n+1) = \sum_{k=0}^{n} \binom{n}{k} B(k).$$

Using this identity, we obtain that

$$B'(x) = \sum_{n=1}^{\infty} B(n) \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} B(n+1) \frac{x^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} B(k) \frac{x^n}{n!} = B(x) e^x.$$

Thus, $e^x = \big(\ln B(x) \big)'$. Using the Fundamental Theorem of Calculus, we obtain that $\ln B(x) = e^x + C$ for some real constant C. Since $B(0) = 1$, the constant C equals -1 and, therefore, $B(x) = e^{e^x - 1}$.

We can also solve sequences defined recursively by using exponential generating functions; indeed, for this purpose, exponential generating functions are sometimes more convenient. This is illustrated by the following example.

Example 7. Let us find an explicit formula for the exponential generating function of the sequence $(a_n)_{n \ge 0}$ if $a_0 = 1$ and $a_{n+1} = (n+1)a_n + 2$. Let $A(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$ be the exponential generating function of $(a_n)_{n \ge 0}$. From $a_{n+1} = (n+1)a_n + 2$, we obtain that

$$\begin{aligned} A(x) - 1 &= \sum_{n=0}^{\infty} a_{n+1} \frac{x^{n+1}}{(n+1)!} = \sum_{n=0}^{\infty} (n+1) a_n \frac{x^{n+1}}{(n+1)!} + 2 \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} \\ &= x \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} + 2 \bigg(-1 + \sum_{n=0}^{\infty} \frac{x^n}{n!} \bigg) = xA(x) + 2\big(e^x - 1\big). \end{aligned}$$

Therefore $A(x)(1-x) = 2(e^x - 1) + 1 = 2e^x - 1$, and so $A(x) = \frac{2e^x-1}{1-x}$. As a result,

$$\begin{aligned}
\sum_{n=0}^{\infty} a_n \frac{x^n}{n!} &= A(x) = 2\frac{1}{1-x}e^x - \frac{1}{1-x} \\
&= 2\left(\sum_{n=0}^{\infty} n!\frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right) - \sum_{n=0}^{\infty} n!\frac{x^n}{n!} \\
&= 2\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}k!\right)\frac{x^n}{n!} - \sum_{n=0}^{\infty} n!\frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty} n!\left(\sum_{k=0}^{n}\frac{2}{(n-k)!} - 1\right)\frac{x^n}{n!}.
\end{aligned}$$

Hence

$$a_n = n!\left(\sum_{k=0}^{n}\frac{2}{(n-k)!} - 1\right).$$

Practice Exercises

Exercise 1. [1, Exercise 8.20] *Let a_n be the number of permutations in S_n whose square is the identity.*

(1) *Prove combinatorially that $a_{n+1} = a_n + na_{n-1}$.*
(2) *Find an explicit formula for the exponential generating function of $(a_n)_{n\geq 0}$.*

Exercise 2. [1, Exercise 8.32] *Consider the sequence $(a_n)_{n\geq 0}$, where $a_0 = a_1 = 1$ and $a_n = na_{n-1} + n(n-1)a_{n-2}$ for every $n \geq 2$. Find an explicit formula for the exponential generating function of the sequence $(a_n)_{n\geq 0}$.*

References

[1] M. Bóna: *A Walk Through Combinatorics: An Introduction to Enumeration and Graph Theory* (Fourth Edition), World Scientific, New Jersey, 2017.

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