MIT 18.211: COMBINATORIAL ANALYSIS

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Lecture 17: Generating Functions III: Back to Partitions and The Composition Theorem

Back to Partitions. Fix $k \in \mathbb{N}$ and, for every $n \in \mathbb{N}$, let $p_{\le k}(n)$ be the number of partition of n which each part having size at most k. Let us find the generating function of the sequence $(p_{\le k}(n))_{n \in \mathbb{N}_0}$.

Proposition 1. *The following identity holds:*

$$\sum_{n=0}^{\infty} p_{\le k}(n) x^n = \prod_{i=1}^{k} \frac{1}{1-x^i} \qquad \textit{for every } k \in \mathbb{N}. \tag{0.1}$$

$$\sum_{n=0}^{\infty} p(n) x^n = \prod_{i=1}^{\infty} \frac{1}{1-x^i}. \tag{0.2}$$

Proof. For each $n \in \mathbb{N}_0$, set $R_n := \{(c_1, \dots, c_k) \in \mathbb{N}_0 \mid c_1 + 2c_2 + \dots + kc_k = n\}$. Observe that R_n is in bijection with the set $P_{\le k}(n)$ consisting of all partitions of n with large parts at most k; indeed, for each $(c_1, \dots, c_k) \in R_n$, we obtain a partition of n with c_i parts of size i. Since $(1-x^i)^{-1} = \sum_{m=0}^{\infty} x^{mi}$ for every $i \in \mathbb{N}$, we see that

$$\begin{aligned}
\prod_{i=1}^{k} \frac{1}{1-x^i} &= \prod_{i=1}^{k} (1 + x^i + x^{2i} + x^{3i} + \cdots) \\
&= \sum_{n \in \mathbb{N}_0} \sum_{(c_1, \dots, c_k) \in R_n} x^{c_1 + 2c_2 + \cdots + kc_k} = \sum_{n \in \mathbb{N}_0} \sum_{\lambda \in P_{\le k}(n)} x^{|\lambda|} \\
&= \sum_{n=0}^{\infty} |P_{\le k}(n)| x^n = \sum_{n=0}^{\infty} p_{\le k}(n) x^n,
\end{aligned}$$

where ic_i is the degree of the monomial we choose from the i-th factor when multiplying out in the second equality.

We can argue (0.2) in a similar fashion. Let T_n be the set of sequences $(c_k)_{k \in \mathbb{N}}$ whose terms belong to $\mathbb{N}_0$ such that $\sum_{k \in \mathbb{N}} kc_k = n$ (note that this implies that only finitely many terms in the sequence $(c_k)_{k\in\mathbb{N}}$ are different from 0). Now T_n is in bijection with the set $P(n)$ of all partitions of n. Thus,

$$\prod_{i=1}^{\infty}\frac{1}{1-x^i}=\prod_{i=1}^{\infty}(1+x^i+x^{2i}+x^{3i}+\cdots)=\sum_{n\in\mathbb{N}_0}\sum_{\lambda\in P(n)}x^{|\lambda|}=\sum_{n=0}^{\infty}p(n)x^n.$$

□

The Composition Theorem. We have learned before to multiply finitely many generating functions and interpret combinatorially the coefficients of this product. Now we will learn how to compose two generating function and how to interpret combinatorially the coefficients of the resulting generating function. It turns out that we can do so imitating the way we compose functions, and there are many benefits in introducing such a composition.

Definition 2. Let $A(x)$ be the generating function of a sequence $(a_n)_{n\ge 0}$ satisfying $a_0=0$, and let $G(x)$ be the generating function of a sequence $(b_n)_{n\ge 0}$. Then the *composition* of $B(x)$ with $A(x)$ is

$$B(A(x)):=\sum_{n=0}^{\infty}a_nA(x)^n.$$

Remark 3. With notation as in the definition, we have imposed the condition $a_0=0$ because, otherwise, the constant term in the composition would be the sum of infinitely many terms, which is not convenient.

The following result gives an interpretation to the composition of generating functions.

Theorem 4. *For each $n\in\mathbb{N}_0$, let a_n be the number of ways to build certain α-structure on an n-set, and let b_n be the number of ways to build a certain β-structure on an n-set. Assume that $a_0=0$ and $b_0=1$. Now let f_n be the number of ways to split $[n]$ into nonempty sub-intervals, build an α-structure on each sub-interval, and then build a β-structure on the set consisting of all such sub-intervals. Assume that $f_0=1$, and let $A(x)$, $B(x)$, and $F(x)$ denote the generating functions of $(a_n)_{n\ge 0}$, $(b_n)_{n\ge 0}$, and $(f_n)_{n\ge 0}$, respectively. Then $F(x)=B(A(x))$.*

Proof. For each $n\in\mathbb{N}$, set $C_k(n):=\{(c_1,\dots,c_k)\in\mathbb{N}^k\mid\sum_{i=1}^{k}c_i=n\}$. Note that the elements of $C_k(n)$ are compositions of n. In addition, observe that for every $n\in\mathbb{N}$,

$$f_n=\sum_{k=1}^{\infty}b_k\sum_{(c_1,\dots,c_k)\in C_k(n)}a_{c_1}\cdots a_{c_k}.$$

because $C_k(n)$ is empty when $k > n$. Therefore

$$\begin{aligned} F(x) = 1 + \sum_{n=1}^{\infty} f_n x^n &= 1 + \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} b_k \sum_{(c_1,\dots,c_k)\in C(n)} a_{c_1} \cdots a_{c_k} \right) x^n \\ &= 1 + \sum_{k=1}^{\infty} b_k \sum_{n=1}^{\infty} \left(\sum_{(c_1,\dots,c_k)\in C(n)} a_{c_1} \cdots a_{c_k} \right) x^n = 1 + \sum_{k=1}^{\infty} b_k A(x)^k \\ &= B(A(x)). \end{aligned}$$

□

Corollary 5. *For each $n \in \mathbb{N}_0$, let a_n be the number of ways to build certain α-structure on an n-set and assume that $a_0 = 0$. Let f_n be the number of ways to split $[n]$ into nonempty sub-intervals, and then build an α-structure on each sub-interval. Assume that $f_0 = 1$, and let $A(x)$ and $F(x)$ denote the generating functions of $(a_n)_{n\geq 0}$ and $(f_n)_{n\geq 0}$, respectively. Then*

$$F(x) = \frac{1}{1 - A(x)}.$$

We conclude with the following example.

Example 6. A total of n students are in line waiting to know whether their final evaluation will be either presenting a project or taking a final exam. As there is no time for every one to present, the instructor splits the line into parts making the students in each resulting segment a team and then select some of these teams to have a final team presentation either in generating functions or in graph theory. In how many ways can the instructor do this? To answer this, we can use Theorem 4 as follows. Once the instructor splits the line into sub-intervals, he turns each sub-interval into a group in $a_n = 1$ way (this is the number of α-structures in the terminology of Theorem 4, so we set $a_0 = 0$). Thus, the generating function of $(a_n)_{n\geq 0}$ is

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=1}^{\infty} x^n = x \sum_{n=0}^{\infty} x^n = \frac{x}{1-x}.$$

Then the instructor assigns to each of the teams one of the following three categories: final exam, generating functions presentation, and graph theory presentation. He can do so in $b_k = 3^k$ different ways, where k is the number of teams (this is the number of β-structures in the terminology of Theorem 4). Then the generating function of $(b_n)_{n\geq 0}$ is

$$B(x) = \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (3x)^n = \frac{1}{1-3x}.$$

Now observe that

$$\begin{aligned} B(A(x)) &= \frac{1}{1-\frac{3x}{1-x}} = \frac{1-x}{1-4x} = \frac{1}{1-4x} - x\frac{1}{1-4x} \\ &= \sum_{n=0}^{\infty} 4^n x^n - \sum_{n=0}^{\infty} 4^n x^{n+1} = 1 + \sum_{n=1}^{\infty} 4^n x^n - \sum_{n=1}^{\infty} 4^{n-1} x^n \\ &= 1 + \sum_{n=1}^{\infty} 3 \cdot 4^{n-1} x^n. \end{aligned}$$

Finally, it follows from Theorem 4 that the desired number is $3 \cdot 4^{n-1}$.

Practice Exercises

Exercise 1. *Let $c(n)$ be the number of self-conjugate partition of n. Find a formula for the generating function of $(c(n))_{n \ge 0}$ that does not involve any summation sign.*

Exercise 2. *Let c_n be the number of compositions of n with an odd number of parts in which each part is at least 2. Find an explicit formula for the generating function of $(c_n)_{n\ge 0}$.*

Exercise 3. *Let c_n be the number of compositions of n in which each part is odd and colored blue, green, or red. Find an explicit formula for the generating function of $(c_n)_{n\ge 0}$.*

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